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RAPID ARITHMETIC

QUICK AND SPECIAL METHODS IN ARITHMETICAL CALCULATION TOGETHER WITH A COLLECTION OF PUZZLES AND CURIOSITIES OF NUMBERS

BY

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NEW YORK
D. VAN NOSTRAND COMPANY
EIGHT WARREN STREET
1922

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PRINTED IN THE UNITED STATES OF AMERICA

PREFACE.

There are many things in arithmetic which receive little or but scant treatment in the ordinary text books. If one method of doing an operation is given, it is considered enough. But it is certainly interesting to know that there are a dozen or more methods of adding, that there are a number of ways of applying the other three primary rules, and to find that it is quite within the reach of anyone to add up two columns simultaneously. The multiplication table for some reason stops abruptly at twelve times; it is not hard to carry it on to or at least towards twenty times. Taking up the question of exponents, it is not going too far to assert that many college graduates do not understand the meaning of a fractional exponent, and as few can tell why any number great or small raised to the zero power is equal to one, when it seems as if it ought to be equal to zero.

The multiplication table can be made to give the most curious relations between its constituent figures and quantities, and of these only a part are given here, for one can play a regular game of solitaire with the multiplication table.

The book now in the reader's hands has been a wonderfully interesting work in its preparation. The sources of information and the authorities appealed to were many and in a number of cases were little known. The collection of the matter presented here was a sort of gleaning, picking

up what others had left. It was also a selective task, the accumulating of the best from many sources.

A reference to the table of contents will show that this preface tells only of a small part of what is to be found in the book. In a certain sense this work is a supplement to the ordinary text book of arithmetic. But it is more than that; it will be found of practical application in real work, for by applying the methods to be found in its pages a greater command of arithmetical operations will be acquired and quick ways of calculating will result.

Much that is amusing in the way of oddities and recreations in the science of numbers will be met with in its pages.

The compiler hopes that his mixture of the useful with the lighter phases of his subject will prove acceptable to the reader.

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CHAPTER I

NOTATION AND SIGNS

INTRODUCTORY

Arabic Notation

The distinguishing feature of the Arabic notation, used all over the civilized world, is the fixing of values by position. In any integral number the decimal point is assumed to be placed at the right of the number—the right hand digit of the number is taken as being on the left of the omitted decimal point. The digit placed here expresses the number of units in the quantity, after the tens, hundreds and other higher decimals have been taken out. Then the next number to the left gives the tens, the next the hundreds and so on.

All this is simple and elementary. But suppose there were no position system of fixing values; then we would be as badly off as were the old Romans, with their clumsy system of literal notation. To express the number eight hundred and eighty eight, if there were no place or position system, we should have to write, 8 hundreds, 8 tens and 8, instead of simply, 888.

Now compare the number of characters required to write this number in Roman and in Arabic systems. In Roman notation it runs: DCCCLXXXVIII, twelve letters as against the three numerals expressing the same thing in Arabic notation—888.

[illegible]

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[illegible] places
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Any number whatever raised to the zero power is equal to 1.

It is the only number all of whose powers are equal to itself. 1^2, 1^3 and any power of 1 is equal to 1.

The powers exceeding 1, of all numbers greater than 1, are greater than the numbers so raised. $2^2 = 4$, $3^2 = 9$; the power is always greater than the original number.

The powers exceeding 1, of all quantities smaller than 1 are smaller than the quantities. $(1/2)^2 = 1/4$, $(1/3)^3 = 1/27$.

The product obtained by multiplying together any numbers, each greater than 1, gives a quantity larger than either of the multipliers; if the quantities are less than 1 their product will be a smaller quantity than either of them.

Thus the number 1 stands at a dividing point and quantities greater than unity or 1, present characteristics differing from those of quantities less in value than 1.

Signs in Arithmetic

Signs are used in arithmetic as a sort of short hand to indicate operations to be performed on the numbers or quantities in the operations.

In arithmetic number and quantity are generally to be taken as synonymous.

The meaning of a sign is expressed in English or in a few cases in Latin; the latter gives a shorter expression.

The sign of addition is a rectangular cross, with its members respectively horizontal and vertical, placed between the two numbers to be added. It is almost always expressed as "plus," the Latin word for "more;" it is perfectly proper to render it, "added to." Where more

than two numbers are to be added, a plus sign is placed between each; 2 + 2 indicates the addition of 2 to 2; 2 + 2 + 3 + 4 indicates the addition of 2, 2, 3, and 4; the first addition giving a sum of 4, the second a sum of 11.

The result of a completed addition is the sum of the numbers or quantities added. The operation is not correctly called a sum.

The sign of subtraction is a horizontal bar or dash; it is placed between numbers, the last number to be subtracted from the first one. It is usually expressed as "minus," the Latin word for "less," we may say correctly to express it, "diminished by," but minus is always used. 5—4 reads 5 minus 4, and means four subtracted from five. It will be observed that in stating such a subtraction the larger number is always placed first as 5—4, five minus four or 6—3, six diminished by three. The latter expression is not in use to any extent.

The quantity from which a quantity is subtracted is called the minuend, from the Latin, meaning "to be diminished"; the quantity which is subtracted is called the subtrahend, from the Latin, meaning "to be subtracted"; the result of the operation is called the remainder or difference.

What has been said about the larger quantity being placed first, in the expression of subtraction, refers to arithmetic, not to algebra.

Multiplication is indicated by a diagonal cross, placed between the numbers; 4×5 indicates 4 multiplied by 5. In continued multiplication the sign must be placed between each pair of quantities; $4 \times 5 \times 6$ indicates 4 multiplied by 5 multiplied by 6, whose product is 120.

Sometimes a period is used as the sign of multipli-

cation. It is liable to cause confusion, by being taken for the decimal point, yet it is very frequently employed.

If multiplications are set down in full, the upper quantity is called the multiplicand, from the Latin, meaning "to be multiplied"; the lower quantity is called the multiplier; and when the operation is completed the result is called the product. There is no reason for placing the multiplicand above the multiplier; they can change position and rôles without affecting the result of the calculation.

Division is indicated in several ways. One is by a horizontal bar or to save space a diagonal one. The number above the bar is called the dividend, from the Latin, meaning "to be divided"; the number with which it is to be divided is placed below the bar; it is called the divisor. Thus $\frac{6}{3}$, or what is the same thing, $6/3$ indicates 6 divided by 3. The result of a division is called the quotient, from the Latin, meaning "how often."

Another sign of division is possibly derived from this one; it is a horizontal bar with one point over its center and one below its center. $6 \div 3$ indicates 6 divided by 3.

The colon, :, is a sign of ratio and hence of division, but is not used, simply to indicate division, except in special cases.

The horizontal or oblique bar is not always admitted to be the sign of division; the claim is sometimes made that there is a difference between such expressions as $2 \div 4$ and $2/4$. The latter is taken as indicating a fraction only. But if we take such an expression as a/b, it is hard to see how it can be expressed except as "a divided by b." In the case of numbers the alternative fractional nomenclature is always available, as "two fourths" in the expression or fraction, $2/4$.

Whatever name is given it, $2/4$ means and indicates the division of 2 by 4.

The indication of the higher power of a number is a small figure placed above and to the right; it is called an exponent; 4^2 means the square of 4, which is 16; 5^3 means the third power or cube of 5, which is 125. The little figures, 2 and 3, are exponents, in these instances, of 4 and of 5 respectively.

The term "square" is an abbreviated way of expressing the second power; the term "cube" is the same for the third power; there are no other abbreviations of power expressions.

The radical sign indicates the root of a number. By itself it indicates the square root; if any other root is to be indicated the exponent is placed over it to the left. $\sqrt{16}$ means the square root of 16 which is 4; $\sqrt[4]{16}$ means the fourth root of 16, which is 2.

When there are several numbers required to express a quantity the combination is called an "expression." $2 + 3$ and $3 + 5$ are expressions.

The sign of equality is a pair of horizontal bars parallel to one another, $=$; they read "equal" or "equals." Thus to express the result of adding two quantities, say 2 and 3, we would write, $2 + 3 = 5$, which reads 2 plus 3 equal 5.

A statement such as the above, affirming the equality of two quantities or expressions, is called an equation and sometimes a formula.

Inequality is indicated by a V-shaped symbol placed on its side; the quantity next the apex is stated to be the smaller. $7 > 2$ means that 7 is greater than 2, or that 2 is less than 7. Both come to the same thing; the sign can face the other way; $2 < 7$ means that 2 is less than 7.

Two colons, placed one after the other, are the central sign of a proportion and are read "as," while in the same proportion the single colon is read "is to"; thus 2:4::4:8 reads 2 is to 4 as 4 is to 8. Sometimes the equality sign, =, is used as the central symbol. If this is done in the above proportion we would have: 2:4=4:8.

Taking the colon as the sign of division our last proportion would read 2 divided by 4 equal 4 divided by 8, which is perfectly correct, and expresses the relation existing between the members of a proportion. The double colon is never taken as a sign of equality, though it might be.

A parenthesis expresses the fact that the quantities within the parenthesis are to be taken as a group and to be treated as a single or individual quantity. $7 - (2 + 3)$ means that the sum of 2 and 3 is to be subtracted from 7; this gives a remainder of 2. The same numbers without the parenthesis, $7 - 2 + 3$ tell us that 2 is to be subtracted from 7 and that 3 is to be added; the result is 8. The parenthesis brings about a totally different result.

Multiplication and division signs take precedence of addition and subtraction signs; the first two signs unite the numbers between which they are placed, as if they were in a parenthesis. Thus $12 - 10 \div 2$ indicates that 10 is to be divided by 2 and the quotient is to be subtracted from 12; this gives 7, as the result. In like manner $4 + 6 \times 3$ indicates that 3 times 6 are to be added to 4, which gives 22; 6×3 act as if in a parenthesis.

Decimal Fractions

A decimal fraction in the broad sense is a fraction whose denominator is a power of 10; $1/10$, $2/100$ and

$\frac{5}{1000}$ are decimal fractions under this definition. Generally the term is restricted to the writing the same class of fractions on the line, using the decimal point.

To do this the numerator of the fraction is written to the right of the decimal point and the denominator, as such, is omitted and is expressed by the relation of the numerator to the decimal point, and this relation is fixed by the use of ciphers, or sometimes simply by the digits in the numerator. Thus if the digits in the numerator are too few to give the proper position or distance from the decimal point to the numerator, ciphers are placed to the right of the point between it and the numerator.

The number of digits or ciphers less one in the denominator of a decimal vulgar fraction gives the number of places in the decimal fraction. $\frac{1}{10}$ is written .1, because there are two digits in the denominator; $\frac{3}{100}$ is written .03, because there are three digits in the denominator.

The Arithmetical Complement

Complement means anything which fills up or completes. If a number is referred to the multiple of 10 immediately above it, the difference between it and that multiple of 10 is its complement. On this basis 2 would be the complement of 18, because the multiple of 10 next to 18 is 20.

Often the higher number to which it is referred is the next higher power of 10.

On this basis the complement of 18 would be 82, for the next power of 10, above 18, is 100.

The arithmetical complement of a decimal fraction is the difference between it and unity or 1.

Thus the arithmetical complement of .55 is .45. This

complement is constantly employed in trigonometrical calculations. It is most easily obtained by subtracting the right hand digit of the decimal from 10 and the other digits from 9. Suppose we want the arithmetical complement of .4658; we proceed as follows: $10-8=2$; this is the right hand digit of the complement. Then we go on and subtract from 9; $9-5$, $9-6$ and $9-4$ respectively; the result is .5342. The sum of a decimal and its arithmetical complement is always unity or 1.

Placing Symbols and Numbers

To put a symbol or number in front or to the left of another is to prefix; to put a symbol or number to the right of a number is to annex.

If a period is prefixed to 55 it converts it into a decimal, .55. If a 5 is annexed to 55 it gives 555. It is important to distinguish between annexing and adding.

A number to which a sign is affixed is said to be affected by that sign. Thus if we write -6, it reads minus 6 and the number, 6, is said to be affected by a minus sign.

The positive sign is never prefixed to a number written alone, yet every positive number may be said to be affected by a positive sign understood.

A number with negative sign prefixed, or affected by a negative sign, is said to be a negative number. A number with no sign prefixed is said to be a positive number.

The use of positive and negative numbers and quantities as such, is generally restricted to algebra.

A number with an exponent is said to be affected by the exponent. Thus the numbers 2 and 5 in the ex-

pressions, 2^3, 5^4 are said to be affected by the exponents, 3 and 4.

The introduction of the plus sign, $+$, and of the minus sign, $-$, is attributed to a German mathematician, Michael Stifel, b.1480, d.1567; the date of their use by him is given as 1544. To him is also credited the radical sign, $\sqrt{\ }$.

The sign of equality, $=$, originated with Robert Recorde, an English mathematician, b. (about) 1500, d.1558. He first used it in 1557.

The sign of multiplication, $\times$, originated with William Oughtred, an English mathematician, b.1574, d.1660. It appeared in his work, Clavis Mathematicae, 1631.

The sign of division, $\div$, is attributed to an English mathematician, Dr. John Pell, b.1610, d.1685.

Addition was originally expressed by writing out the Latin word, plus, by the Italian word, piu, or by the letter, p. Subtraction was indicated by minus, mene, or m. The radical sign succeeded the capital letter, R, as the sign of a root of a number.

The signs of inequality, $>$ and $<$, appear first in a posthumous book of Thomas Harriot, an English mathematician, a contemporary of Oughtred.

The decimal point was introduced by John Napier, (b.1550, d.1617) the famous Scotch mathematician, in the beginning of the seventeenth century.

The decimal system of numbers was introduced into Europe in the tenth century.

CHAPTER II.

ADDITION

Notes on Addition and its Theory

Everyone is supposed to know the multiplication table; this means the ability to name the product of any two numbers within the range of the usual multiplication table, and to do this at once without stopping to think. In the table in question there are, if we count the "one times" part, 144 different multiplications. Some of these are repetitions inverted, such as 3 times 9 and 9 times 3. It is safe to call the products to be remembered 132 in number, as the inverted ones should count as individual products, even if the quantities are identical.

Addition is the first rule of arithmetical operations taught to the child, for numeration is hardly to be called an operation. Yet of all the four basic things in arithmetic, of addition, subtraction, multiplication and division, addition gives the most trouble, is the most subject to error and is the most used of the four. So true is this that while every bank has one or more adding machines, comparatively few consider it worth while to have a multiplying machine. Owing to the extensive introduction of adding machines there is not the same demand for men who are rapid adders, that there once was. But we all have our own adding to do and few have access to an adding machine. It is satisfactory to know that, by proper methods and by analysis of its combinations, the process of addition can be facilitated

to a greater extent than would seem possible. There are a number of different ways of adding up columns of figures, there are a number of special ways in use by individuals, of helping the work along so that many operators have what may be called their own ways of getting at results.

The Addition Table

In the multiplication table there are 144 different multiplications to be remembered and to be known so perfectly that no time or thought is to be given to them when asked or required. In the corresponding addition table there are only 45 additions to be remembered, yet for some reason it is fair to say that they are not as well known in general as are the products of the multiplication table.

The following things are to be noted about the addition of the nine digits to each other, in twos, that is to say in the addition of one digit to another, for all the nine digits.

The total number of such additions is forty five.

Of these additions twenty give single numbers. Such are $2 + 4 = 6$, $3 + 5 = 8$.

Of these additions twenty five give double numbers. Such are $5 + 6 = 11$, $7 + 9 = 16$.

The largest number given by the addition of two digits is eighteen, $9 + 9 = 18$; the figure on the left is 1, this is the figure to be carried when we add; therefore the figure to be carried in addition is never increased in an addition to any number of a single digit by more than 1.

To illustrate this statement suppose 6, 7, 8 and 9 are to be added together. 6 and 7 are 13; this gives 1 to carry. Then comes 3 and 8 giving 11 to which is added the 1

carried from the first addition, giving 21, so that there is 2 to carry. Next 1 is added to 9 giving, with the 2 to carry, 30 as the sum of the four numbers.

There was first 1 to carry, it could not possibly be more than 1. The next number to be carried, due to the addition this time of three figures was 2, one more than the first figure carried, which was 1. Then when the addition of the four was completed the figure in the tens' place was one more than the last figure carried, it was 3.

This leads to the following conclusion: When a column of single digits is added up the successive left hand figures in the ten places can never exceed each other by more than 1 at a time; successive figures in the ten places may be the same.

In the following additions the separate additions are given at the right hand side of the column of amounts to be added.

a		*b*		*c*		*d*	
9	50	1	22	9	32	9	40
8	41	2	21	8	23	8	31
9	33	2	19	7	15	7	23
7	24	8	17	8		8	16
8	17	9		—		8	
9		—		32		—	
—		22				40	
50							

Column *a* is made up of such figures as involve the carrying of an additional 1 for each adding; this is the most that can be carried and it will be observed that the successive sums are, as regards their figures in the ten places, successively greater by 1, so that the ten place

figures run 1, 2, 3, 4 and 5. They never can increase more rapidly than this.

Next take column *b*. Here small figures are added; the increase of the ten place figures is slower; they run 1, 1, 2 and 2.

In column *c* four figures are added giving the maximum difference of the ten place figures, in this case 1, 2 and 3. In column *d* the same four figures are taken but an 8 is placed below them. On addition we again have the regular maximum succession of ten place figures, but in this case running up to 4, because of the extra 8 which has been added in.

Carrying One

Now comes the subject of what additions do not give, and of which do give "one to carry"; both kinds are given here.

11111111	222222	3333	44
12345678	234567	3456	45
23456789	456789	6789	89

These are the twenty additions which do not give "one to carry"; next come the twenty five which do involve the carrying of 1.

1	2	2	3	3	3	4	4	4	4
9	8	9	7	8	9	6	7	8	9
10	10	11	10	11	12	10	11	12	13

5	5	5	5	5	6	6	6	6	7	7	7	8	8	9
5	6	7	8	9	6	7	8	9	7	8	9	8	9	9
10	11	12	13	14	12	13	14	15	14	15	16	16	17	18

How to Add

In the modern teaching of reading the effort is made to teach the pupils to learn to know words without the process of spelling. In adding numbers they should be added without naming; thus in adding the figures in column *a*, page 13, you should say to yourself in quick succession 9, 17, 24, 33, 41, 50; do not say to yourself 9 and 8 are 17 and 7 makes 24 and so on.

It is interesting to use the tables just given to test your quickness at addition; if you cannot give the additions of the tables without hesitation and almost without thinking, fast addition and even accurate addition to say the least will be difficult.

Different Ways of Adding

Addition may be done column by column and one digit at a time; this is probably the most usual way of doing it. It is the most obvious and the simplest but is also the slowest. There are two ways of doing it—from above down or from the bottom of the column upwards. To prove the accuracy of the operation it is well to do it first one way and then the other.

Bookkeeper's Addition

A minor variation which can be used to advantage is to write down the sum of each column separately, one sum under the other and each successive sum set one space to the left; then a subsidiary addition gives the total. This is done here:

9938	Addition of first column	30
7827	" " second column	8
4119	" " third column	26
6826	" " fourth column	26
28710	Total	28710

In the left hand addition it is added up in the regular way; in the right hand addition the way just described is carried out.

Group Addition

The different additions of two digits give only 17 different numbers, so it is a very easy matter to know them. It is not easy to give them without stopping to think. When perfectly known they lead to another way of adding, the first step of group adding.

This method consists in adding two or more numbers at a time to two or more others in the vertical columns. Thus the following addition would be done by at once seeing in the 8 and 7 the number 15 and in the 9 and 3 above them the number 12; then 12 and 15 would be added, giving 27. This may or may not involve adding double numbers to each other, but this method is made simple by the fact that the sum of two numbers can never have anything higher than 1 in the ten place and nearly half the time will not have even that.

3	
9	12
7	
8	15
–	—
	27

When several columns are to be added by grouping, there will generally be a number to be carried; this may be added to the first group of the next column, or the whole operation may be treated as is shown on later pages.

If the direct adding of such numbers as 15 and 12 to each other is too difficult, then for rapid work proceed as follows: In the last example add 15 to 10 giving 25; then to this 25 add the missing 2 of the 12, for you have already added its 10, reading it off to yourself in this way: 15, 25, 27. This system makes it as simple as ordinary adding. The process of group adding is easier and better in every way than the direct single column way.

Index Adding

Two similar methods of addition are next given in the examples at the side of the pages, which while simple enough are of some value. Every different way of adding up a column of figures, if it has no other value, is of use as a way of testing the accuracy of the operation or, as it is usually put, of "proving it."

8 7
9
7 8
3
2
6
8 5
7
6 5
3
6
—
65

Referring to the left hand column it has been added up as follows: Starting at the bottom the figures have been added until they approach 20 so closely that the addition of the next digit above would give 20 or a sum exceeding 20. At this point the last figure of the sum reached is written to the side of the column. A new addition, one with no reference to the one just done, is started and carried on until 20 is again approached; the terminal digit is written and this is repeated until the top is reached. If the top addition has no ten place figure in it, if it is less than 10, nothing is written at the side. Then all the figures at the side are added up, if there are any top figures above the last or uppermost key-figure these are added in also, and in front of the sum is written a number for the tens equal to the number of key-figures written at the side, increased by anything to be carried from the addition of the key-figures.

In the left hand column the lowest three figures added up give 15. The figure next above them is 7, this is not to be added in because the sum would exceed 20. Therefore 5 is written on the side. Starting anew, we add 7 and 8 giving 15; here we must stop for the next figure is 6 which added in would give 21. So stopping here 5 is written on the side. In this way we go on until the top

is reached. As the last or highest figures of the column give a number in excess of 10 the figure 7 of their sum, 17, is written. Now the key-figures are added up; they give 25. We write down the 5 and carrying the 2 add to it 4, or 1 for each key-number. There are four key-numbers, so the total is for the tens, $2 + 4$, which gives 6, and the sum of the whole is 65.

Suppose now that there had been no eight at the top of the column and that after the last key-number had been written only the 9 had been left; then there would have been only three key-numbers, 5, 5 and 8 making 18; to these would have been added the remaining 9, at what would then have been the top of the column; this would have given 27; then there would have been only 2 to carry and to go into the ten place in addition to the 3 of the key-numbers; we would have a total of 57, which would be the true sum of the column without the topmost figure 8.

Period Addition

8 .
9 .
7 .
3
2 .
6
8
7 .
6 .
3
6
—
65

The next process is done in a similar way. The addition of the figures is continuous this time right up to the top, except that when the number, just as before, approaches 20 a dot is placed at the side, and the addition goes on but with omission of the tens. Thus after the third figure from the bottom is reached we make a dot and go on with 5 as a starting figure. Then we have $5 + 7$; this gives 12. The next figure is 8, so we put a dot at the side and starting with 2 add it to the figures above it, 8, 6, and 2, find that a dot goes here where the total is 18; then this 8 is added to the 3 and 7 above it and another dot is put down and for the final figures we have 17 for the next to the last

dot and carrying the 7 the top figure gives $7 + 8$ or 15. Here is the last dot and the unit figure 5 to be written down, while for the tens count the dots. There are six so our tens are 6 and the number is 65.

Just as before, if there is no number exceeding ten in the last addition, the numbers are simply added in to make up the units. As in the first of the two methods, there may be 1 to carry at the top.

Addition by Combination

It is sometimes advised that the combinations of numbers adding up to ten be memorized. Such are 3, 3, 4; 1, 3, 6; 2, 3, 5. The two figure combinations are so well known that it is to be presumed that everyone knows them. Anyone can write out the different combinations and study them. It is also recommended to learn the combinations of more figures, such as the eight combinations of four figures which give 20. There are nine combinations of four figures which give 30. The higher combinations are of little use, for the more numbers involved in a combination the rarer it is. The two and three figure combinations are the most useful.

Every such combination which can be memorized is an addition to one's faculties in doing group adding. Group adding should not be confined to sets of only two numbers. All these ways contribute to group adding. It will often be found that expert adders have their own favorite ways of grouping.

Addition by Average Multiplication

When numbers occur, whose middle figure is the average of the lot, multiplying it by the number of figures will give the sum. Suppose 5, 4 and 3 are to be added; 4

is the average of the three figures; therefore $4 \times 3 = 12$ is the sum. Such combinations as this are to be watched for and made use of.

Addition by Multiplication

The following is another way of adding up a column of single figures. By adding and subtracting the figures are all reduced to the same, and a simple multiplication gives the addition provided the same amounts were added as were subtracted. Otherwise a correction must be made. Some examples follow.

$$
\begin{array}{lcl}
9 - 1 & = & 8 \\
9 - 1 & = & 8 \\
8 & = & 8 \\
7 + 1 & = & 8 \\
7 + 1 & = & 8 \\
\hline
40 & & 40
\end{array}
\qquad\qquad
\begin{array}{lccr}
9 - 2 & = & & 7 \\
6 + 1 & = & & 7 \\
3 + 4 & = & & 7 \\
4 + 3 & = & & 7 \\
8 - 1 & = & & 7 \\
\hline
30 & & 5 & 35 \\
 & & & 5 \\
\cline{4-4}
 & & & 30
\end{array}
$$

The idea is to substitute a simple multiplication for an addition. In the second case as 8 has been added and 3 subtracted, the net result is an addition of 5, which has to be subtracted from 35 to give the answer.

Decimal Addition

The next method may be termed decimal adding. To all of the quantities add such quantities as will make them multiples of 10 or of 100. From the last of the original quantities subtract the sum of the increments or

amounts added. A simplified addition gives the sum required.

Add 9, 7, and 4.	Add 97, 89, and 49.
$9+1=10$	$97+3=100$
$7+3=10$	$89+11=100$
$4-4=0$	$49-14=35$
20	235

In the first example the increments are 1 and 3, their sum is 4, and this is subtracted from the last of the original quantities, which is also 4. The sum of the numbers in the right hand column gives the sum of the original numbers. In the second example 14 is the sum of the increments, and is subtracted from 49, the last of the original numbers. The sum of the right hand column gives the answer.

Two and Three Column Addition

There are ninety combinations of double numbers, 10, 11, 12 and so on up to 99. If the sum of any two of these can be given without hesitation at sight, then two columns can be added simultaneously. This implies double speed, and there are many computers who can do this. Some can add three columns simultaneously.

The addition of two columns at once can be done by the following method by anyone. It is a method well worth acquirement.

The bottom or top number of the two columns is added to the tens of the number next it, and then the units of the next number are added; this gives the sum of the two. Then the tens of the third number are added and then

the units of the third number giving the sum of the three. This is continued to the end of the columns.

29
34
71
88
222

To apply this method to the example here given proceed as follows: Add 70 to 88; this gives 158. To this add the 1 of the 71, giving 159. To 159 add 30, giving 189, and to this add the 4 of the 34 giving 193. To 193 add 20, giving 213 and to this add the 9 of the 29 and the total is 222.

A variation can be used and the units can be added in before the tens; thus $88 + 1 = 89$. This takes care of the unit figure of the 71. Then add 70, which makes 159. $159 + 4 = 163$ and adding the 30 of 34 to this gives 193. Then $193 + 9 = 202$ and 20 added to this gives the total 222 as before.

Three columns in width can be added by an extension of the same method. Thus to add 957 to 875 if we start with the hundreds it goes as follows: $957 + 800 = 1757$; $1757 + 70 = 1827$; $1827 + 5 = 1832$.

The three column method is sometimes said to be only useful within the limits of a limited range of numbers; it is good practice however. The two number adding is quite practical for much longer rows of numbers. An example of three column addition follows.

Add the following:

541	$1042 + 500 = 1542$
237	$1002 + 40 = 1042$
764	$1001 + 1 = 1002$
	$801 + 200 = 1001$
1542	$771 + 30 = 801$
	$764 + 7 = 771$

As the addition was started at the bottom of the columns the sum is reached at the top.

Left Hand Addition

When a number of columns are to be added the addition may commence with the left hand column at the bottom. When completed the adjacent figure of the next column is annexed, not added, to it and the addition is carried right down the second column. The addition or sum of the two columns is written down at the bottom and the next two columns are treated in the same way. If their sum includes only two digits they are set down to the right of the sum of the first two columns; if they comprise three or more digits they are set down one line below the others and with all except the two right hand digits under the right hand digits of the upper sum. The two are then added if there are only the four columns; if there are more, the same process is carried out. The one thing to be kept in mind is that the sum of the first, or right hand column is not added to the second, but is treated as the tens and hundreds of a quantity to which the second column supplies the units.

$$\begin{array}{r} 1798 \\ 9788 \\ 8967 \\ 7899 \\ \hline 281 \\ 352 \\ \hline 28452 \end{array}$$

In the example the first column adds up to 25; assume that we are adding from below upwards. Then to 25 we annex the 7, which is the top figure of the second column and go down with the addition, 257, 264, 273, 281. The last is the sum of the first and second columns counting from the left and is written down below the line. The same is done for the next two columns; we get 32 for the next to the last column; we annex 8 giving 328, and adding as before we get 328, 336, 343, 352. As this has more than two digits it is written a line below the other addition, its left hand figure under the right hand figure of the other sum. The two are then added.

An interesting variation on the above method is the following. Take the same example to be added.

We start as before at the lower left hand figure and add up the left hand row, which gives 25. Then as before annexing the 7 at the top of the next row which gives 257 as before we finish the row and get 281. Here the difference comes in. We put down 28 only and annex the 9 at the foot of the next column, which gives 19. Then proceeding as before when we reach the top of the third column from the left, we have 42. To this annex the 8 at the top of the right hand column, making 428, and then going down the column to the foot we get as the final sum of the last operation 452, which is put down directly in place. There is here no subsidiary addition; the whole sum is put down in one row, completing the addition. (See example *a*.)

Had there been more than four columns the numbers 28 would have first been put down; then when the second addition was completed the two left hand numbers 45 would be put down, and the 2 would be carried on and annexed to the next column. (See example *b*.)

(*a*)	(*b*)
1798	17986
9788	97882
8967	89675
7899	78991
28	28
452	45
28452	34
	284534

In practice the additions in the last two examples would be written at once on the same line; they are here placed on different lines to illustrate and explain the method.

Addition without Carrying from Column to Column

A very convenient way of adding several columns is the following. The first column, the right hand one, is added up and the sum is written down, under the first and, if of two figures, with its second figure under the second column. If there are three figures in the sum of the first column then its third figure goes under the third column. Next the third column is added up and its sum is set down with its first figure under the third column, its second figure under the fourth column and so on. Suppose there are four columns to be added. Then returning to the second column it is added up and its sum is put down on the same principle on the line below the sums already set down. Finally the fourth column is added up and treated in the same way. A final addition of the two rows of figures gives the total.

$$\begin{array}{r}
8984 \\
8435 \\
9917 \\
\hline
2216 \\
2512 \\
\hline
27336
\end{array}$$

In the example the first two figures of the upper line, 16, are the sum of the first column—$4 + 5 + 7$; the next pair are the sum of the third column—$9 + 4 + 9$. Going down to the next row 12 is the sum of the second column—$8 + 3 + 1$, and 25 is the sum of the fourth column—$8 + 8 + 9$.

If there are three figures in any sum they are set down as described and the only effect is that one more

row is required. Three figures in any sum can only occur when the column is more than twelve figures high. Suppose that the first column of four columns added up 116, the third, 322, the second, 312 and the fourth 125. They would be put down thus:

```
   116
  322
 312
125
------
160536
```

In such a case as this it would be much better to set them down in regular order. The method only applies when practically each sum is less than 100; when each sum contains less than three figures.

Another method of adding is shown in the next example.

```
 382925
 399098
 976879
-------
1547781
 21112
-------
1758901
```

Starting at the left the column is added up giving 15, which is put down as shown. The next column is added up, it gives 24; this is put down, the 2 under the 5 and the 4 next to the 5 on the same line. The other columns are treated in the same way; their sums are 17, 17, 18 and 21. All are put down obliquely on two lines and the final addition gives the sum of the whole.

Decimalized Addition

For additions of two small quantities a convenient way is to make a decimal out of one of them by addition or subtraction as the case may be. The other number is treated in the exact reverse way, and the two are added. Suppose 97 is to be added to 86; add 3 to the 97 making it 100; treat the 86 in the reverse way and subtract 3 from it, which gives 83, and the sum is instantly seen to be 183. If we only add the 3 to the first number let the other alone and add, the subtraction of 3 from the sum gives the desired total.

$$\begin{aligned} 97+3 &= 100 \\ 86-3 &= 83 \\ \hline & \ 183 \end{aligned}$$

Carrying out the same method to a fuller development, leads to the following method of performing addition. Reduce each number to a decimal by adding or subtracting as you please; add the decimals and add or subtract the increments, adding such as you subtracted and subtracting those you added. Thus to add 341, 896 and 302. Add 59 to the first number, add 4 to the next one and subtract 2 from the last. This gives the result shown; the unchanged figures are first given, then the new ones and then the increments to be added or subtracted.

341	+ 59	400
896	+ 4	900
302	− 2	200
1539	61	1600 − 61 = 1539

As 59 and 4 have been added to decimalize the numbers they have to be subtracted from the sum of the decimals, and as 2 has been subtracted it has to be added. So we say 59 and 4 are 63, to be subtracted and 63 less 2 gives 61, the net sum to be subtracted.

Sight Reading in Addition

To be able instinctively to name the sum of several figures is what is sometimes called sight reading. Take the two following additions: 23 to be added to 45 and 59 to be added to 75. Treat each case as two separate

additions and to make this clear we shall write the figures with dots to separate them in couples.

```
  2 · 3          5 · 9
  4 · 5          7 · 5
  -----          -----
  6 . 8            1 4
                 1 2
     68          -----
                 1 3 4
```

The eye is supposed to be fixed between the two columns where the dots are; to immediately see a 6 and an 8, making 68, or a 12 and a 14, making 134. Of course the reader will see that this is not a case of straight addition; it is really adding in the one case 60 to 8, and in the other case adding 120 to 14; but it is supposed to be done instinctively. There is a bit of psychology in adding—the calculator who hesitates is lost—the minute you let yourself think and cease to rely on instinct you will fail in rapid work.

Inverted or Left Hand Addition

When two large numbers are to be added to each other the operation is usually begun at the right hand. It is much better to begin at the left hand. This is called inverted or left hand addition.

Add mentally the left hand figures. If the two next to them add up less than 9, write down the sum of the left hand figures. If the two next add up more than 9, carry one, that is increase the sum of the left hand figures by 1 and write it down. If the two next add up 9, then look at those next to them on their right and see how

they add up. If less than 9, write down the addition; if 9 is the sum of the neighboring figures see if their next door numbers are less than 9. If so there is nothing to carry; if over 9 there is 1 to carry; if the sum is 9 then go one more to the right and see if there the sum is less than 9, is equal to 9 or is more than 9, and proceed accordingly. It is perfectly simple when you do it, though it seems a little complicated in the description. The following example will make it clear. Add the following numbers:

$$\begin{array}{r} 7906872 \\ 8293138 \\ \hline 16200010 \end{array}$$

We add mentally, without writing it down, 7 and 8, which are 15. The next numbers on the right add up more than 9, therefore we carry 1 and write down 16. The next numbers are 9 and 2 making 11; next on the right is a sum of 9, the same on its right, and then a third sum of 9 and then a sum of 7 and 3 which is 10; this means the carrying of 1 right down the line giving the three o's and one to carry to be added to the sum of 9 and 2, making it 12, so we write down the 2 having already carried the 1. We now go back to the 7 and 3; next to them on their right are 8 and 2 to be added; these give 10, so we have to carry 1 to the sum of 7 and 3 making it 11; we write down 1, having already carried the other 1; we then write down for the sum of 2 and 8 a cipher, o, because we have already carried the 1.

This is a most complicated example; ordinarily it goes much easier, and with a little practice the addition is written down as if you had it by heart.

It is especially in taking out logarithms that this method is recommended, but it is the best way of adding numbers when only two high. It is not so practical when the column is three or more high.

Complementary Addition

The following is a sort of complementary addition, which is very practical and useful. It is a method of adding two columns at once.

Add the following quantities: 78, 54, 89 and 65.

$78 + 22 = 100$	$54 - 22 = 32$	78
$100 + 32 = 132$	$89 - 68 = 21$	54
$132 + 68 = 200$		89
$200 + 21 = 221$		65
$221 + 65 = 286$		—
		286

Its complement, 22, is added to the first quantity, 78, giving 100. The same complement is subtracted from the next quantity to be added, 54; this gives 32, which is added to 100, giving 132. This is the sum of the first two quantities. To 132 is added its complement, 68, giving 200. This complement, 68, is subtracted from the third of the quantities to be added, 89, and the difference, 21, is added to 200, giving 221. It is as well here to add the last number, 65, directly to 221, which gives the answer, 286.

In cases where the complement is larger than the number next in order, the difference between the two is subtracted from the hundreds quantity. As an example add 62, 11, 23 and 1.

$62 + 38 + 100$	$38 - 11 = 27$ (to be subtracted)	62
$100 - 27 = 73$	$27 - 23 = 4$ (to be subtracted)	11
$73 + 27 = 100$		23
$100 - 4 = 96$		1
$96 + 1 = 97$		—
		97

The complement, 38, of the first number, 62, is larger than the second number. Therefore the process is reversed and the second number is subtracted from the first complement and the difference is subtracted from 100 instead of being added to it. As the complement of 73, obtained by this subtraction is larger than the third number, 23, the latter is subtracted from 27, and the difference, 4, is subtracted from 100, instead of being added to it. This gives 96, to which we add the last figure of the addition, and thus arrive at the total, 97.

After practicing the operations on paper at full length, there will be no difficulty in doing it almost entirely mentally.

CHAPTER III.

SUBTRACTION

Subtraction

Arithmetical subtraction is the subtraction of a smaller number from a larger one. In algebra the reverse may be done, the smaller quantity may be the minuend, and then the remainder will be affected by a minus slgn. But this is outside the scope of arithmetic.

Subtraction may be taken as the simplest of the four elementary processes of arithmetic; ordinarily addition, multiplication or division may be expected to involve more complication. Nevertheless there are a number of modifications of subtraction, a number of ways of doing it, and as we shall see, it may involve complications.

Principles of Subtraction

If we count the cipher, 0, as a digit or number, there will be 100 subtractions of number from number. Thus from 1 we may have to subtract any one of the ten digits: the same is to be said for 2, and as there are ten digits the total number of different subtractions is 10 × 10 or 100.

In some of these subtractions it is necessary to "borrow ten" and to "carry one"; there are forty five subtractions of single digit from single digit in which this has to be done.

To perform the operation of subtraction these operations have to be known perfectly.

Of ordinary subtraction there is nothing to be said. The smaller number, the subtrahend, is usually placed below the larger, the minuend, and the subtraction is done digit by digit. There is little advantage in a simple subtraction, such as this, in doing the work upon two numbers at once. If it is desired to subtract two digits at once the work can be done exactly on the lines described in the preceding chapter for couple adding.

While the subtrahend is usually placed below the minuend, this is not necessary. A computator should be able to subtract upwards as well and as readily as downwards. It often happens that this has to be done, unless the subtraction is rewritten simply to bring the minuend above the subtrahend. One way should be as easy as the other.

Simplified Subtraction

A simplification of plain subtraction is based on the following considerations. It is easier to subtract a multiple of ten from another quantity, than to subtract any other double digit number. It is easier to subtract 30 than to subtract 27. This is the first consideration. The second one is that if numbers are to be subtracted one from the other, the result will be unchanged if we add the same amount to each or if we subtract the same amount from each. Thus $39 - 27 = 12$. Now add 3 to each and subtract; this gives $42 - 30 = 12$. The same quantity has been added to each and the second subtraction gives the same result as the first. Had we subtracted the same number from each the same result would follow. Subtract 7 from each of the original quantities and subtract the quantities so obtained. This

gives $32 - 20 = 12$, the same remainder as in the other two cases.

Decimalized Subtraction

To utilize these principles we select a number to be added to or subtracted from the minuend and subtrahend, which by such addition or subtraction will produce a multiple of ten as the subtrahend. In the preceding paragraph, it will be observed that this was done. Having added or subtracted the two quantities from minuend and subtrahend alike, we will have an exceedingly simple operation to do to obtain the remainder.

Let it be required to subtract 3865 from 9783. The method could be carried out by adding 135 to each number. This is hardly worth while; the regular way is about as easy. But we may divide the quantities into couples and add 5 to each of the right-hand couples and 2 to each of the left-hand couples, and a very simple operation will give the result, the remainder.

a.	*b.*		*c.*
9783	$9783 + 135 = 9918$	$97 + 2 = 99$	$83 + 5 = 88$
3865	$3865 + 135 = 4000$	$38 + 2 = 40$	$65 + 5 = 70$
5918	5918	59	18

The regular method of subtraction is carried out in example *a.*; in *b.* the addition of 135 is employed, and in the third, example *c.*, the subtraction is done in couples, after the addition of 2 and of 5.

The object to be attained is obvious. The subtraction of a decimal number is simple and easy, and such subtraction is brought about by these methods.

Subtraction in Couples

Sometimes when working by couples, there will be one to carry. This introduces no difficulty of any moment. If there is one to carry, 1 is subtracted from the difference of the couple next on the left.

Subtract 3983 from 9765. This is to be done in couples.

Add 1 to the left-hand couples, and add 7 to the right-hand couples. This gives:

9765		98	72
3983		40	90
5782		57	82

In subtracting 90 from 72 we had to borrow 1, so there is one to carry and this is subtracted from the difference of 98 — 40, giving 57 instead of 58.

Sometimes it is better to subtract numbers from quantities of the problem, rather than to add them. If we work in couples, one pair of couples may be treated by subtraction and the other by addition. The last example will be done in couples; in example *a.* 9 will be subtracted from the left-hand couples and 3 from the right-hand couples; in *b.* 1 will be added to the left couples and 3 will be subtracted from the right ones; in *c.* 9 will be subtracted from the left couples and 7 will be added to the right ones.

a.		*b.*		*c.*	
88	62	98	62	88	72
30	80	40	80	30	90
57	82	57	82	57	82

In all cases there was one to carry, so that the second subtraction gives 57 instead of 58. In writing out the result the division into couples is dropped—the remainder is given as 5782.

A variation on the above is, as the case may be, to add to or subtract from only one of the quantities. On subtracting a first remainder is obtained; if a quantity was added to the subtrahend, it has to be added to the first remainder; if it was subtracted from the subtrahend it has to be subtracted from the first remainder. If the minuend is the number added to or subtracted from, the first remainder is treated in the reverse way. A single example will suffice to illustrate this method.

Subtract 287 from 3863. Add 13 to 287 giving 300. This is subtracted from 3863 and the 13 is added to the remainder; $3863 - 300 = 3563$ and $3563 + 13 = 3576$. This is a case of adding. Suppose it was 313 to be subtracted from 3576. Taking 13 from the subtrahend leaves 300; subtracting this from 3576 leaves 3276. From this we have to subtract the 13 giving us the final answer 3263.

Subtraction by Inspection

The subtraction of two-figure numbers can be done by decimalization by simple inspection. Thus to subtract 39 from 73, cast out the 9 and the 3 and say 30 from 70 leaves 40. Now the 3 has to be added and the 9 has to be subtracted, giving a net figure of 9 less 3 or 6 to be subtracted from 40, and we have the answer 34. We give three examples, *a*, *b* and *c*.

a. $73 - 39 = 40 - 9 + 3 = 34$.
b. $97 - 38 = 60 - 8 + 7 = 59$.
c. $98 - 37 = 60 - 7 + 8 = 61$.

Subtraction by Addition

When a salesman makes change he frequently makes it by addition. The same process can be applied to any subtraction. Suppose 281 is to be subtracted from 987. The left hand example shows the regular subtraction and the right hand one the addition method. To do it first write down 281, the subtrahend. Put a line under it and below this line put down 987, the minuend. Then proceed to write above the two numbers a number which added to 281 will give 987; this number is 708, and is the answer.

987	706
281	281
706	987

Now subtract 283 from 931 by addition. We proceed as before except that here we have to carry one in the operation twice. Thus having written 283 over the line and 931 under it, we we say 3 and 8 are 11. We put down the 8 over the 3 and carrying 1, we say, 8 and 1 are 9 and 4 are 13. The 4 is written above the 8 and 1 is carried to the 2; this gives 2 and 1 are 3 and 3 and 6 are 9. Writing the 6 above the 2 we have the remainder, the answer, 648.

648
283
931

Inverted or Left Hand Subtraction

Inverted or left-hand subtraction is so similar to or, better, analogous to inverted addition, that, if the reader has acquired the one, the other follows. It is the better way to subtract and should always be employed in prefence to the usual way.

Subtract 7293138 from 8906872. It is not necessary to write them one above the other; it can be done by simple inspection. We will write it out as usual however, although this is quite unnecessary.

$$\begin{array}{r} 8906872 \\ 7293138 \\ \hline \cancel{1}613734 \end{array}$$

Proceed thus: 7 from 8 leaves 1—there is nothing to carry as the next two figures to the right subtract without 'borrowing ten.' Then 2 from 9 leaves 7, but as the next two figures are 9 from 0, we have to borrow ten—so instead of 7 we write 6. The next figures come regularly until we get to 3 from 7; this is followed on the right by 8 from 2, so instead of 3 from 7 giving 4, as there is 1 to carry we write down 3, and finally write 4 as the remainder of 8 from 12. After a little practice one can write the result off almost as rapidly as if it were being copied. For work in logarithms competent computators do it by simple inspection, writing the result as if copying it.

Subtract 8999 from 9991. Here we look down the line to the right and see that we have in the first and second place to the right, 9 from 9; these involve carrying one because they are preceded by 9 from 1. If we carry 1 to 9 from 9 it gives a remainder of 9 and again 1 to carry. The operation is this: 8 from 9 leaves 1 but as we see that there is 1 to carry the remainder is nothing. 9 from 9 would give nothing but there is 1 to carry, so it becomes 0 from 9 giving 9 as the first figure of the remainder. Then there is another 9 from 9 with 1 carried giving therefore 0 from 9 = 9 as the next figure. We are now at the end and say 9 from 11 gives 2. As we get each figure we write it down—992.

Complement Subtraction

The arithmetical complement of a number is the remainder found by subtracting it from the next highest multiple of 10. It is abbreviated as a. c.

Thus the complement of 1 is $10 - 1 = 9$; the a. c. of 63 is $100 - 63$ or 37. Suppose it is a decimal, then the same rule applies. The a. c. of .361 is $1.000 - 361 = .639$.

The universal way of calculating the a. c. of a number is to subtract its right hand figure from 10 and the others from 9. This avoids the carrying of 1's. The a. c. of 69572 is obtained thus: 6 from 9 gives 3; write this down as the left hand figure. 9 from 9 gives 0; this is put down on the next place to the right; 5 from 9 giving 4 comes next on the right; then 7 from 9 giving 2, and at the extreme right 2 from 10 (not from 9, because it is the right hand end of the row) and the final figure is 8. The a. c. of 69572 is therefore 30428. The next highest multiple of 10 is 10000, or 1 followed by as many ciphers as there are digits in the number. If this is tried once or twice it will be found that the a. c. of a number can be written out just as quickly as the number itself.

The a. c. is used in subtraction; it is especially available where a quantity is to be subtracted from the sum of several others. It is constantly employed in logarithmic calculations.

To subtract by means of the a. c. add the a. c. of the subtrahend and subtract the multiple of ten used in obtaining the a. c. This merely means to throw out a 1 to the left of the a. c. along with any ciphers which come between it and the first number of the sum. To subtract 9872 from 9903 by the a. c. proceed as follows:

	9903
a.c. of 9872	0128
adding	10031
and throwing out 10000	31

The 0 is kept before the a. c. to fix the place of the 1 to be thrown out. The difference or remainder is 31. There is nothing gained by its use in so simple a case. But now take a more complicated case.

Suppose that 1193 is to be subtracted from the sum of 9836, 1072 and 1191. The sum of these three amounts is 12099, and subtracting 1193 from this gives 10906. This involves two operations. To do it by the a. c. write the three numbers to be added one under the other and under them the a. c. of the subtrahend.

	9836
	1072
	1191
a.c. of 1193	8807
adding	20906
throwing out 10000	10906

The throwing out process is simply the subtracting the multiple of 10 used in getting the a. c., or subtracting 1 from the proper decimal place. The proper decimal place is that occupied by the 1 of the multiple of 10 used in getting the a. c; for 1191 we used 10000, so it is 1 in the fifth place which is to be thrown out. Suppose 9991 is to be subtracted from 18931. To get the a. c. of 9991 it is subtracted from 10000. Of course what is really done is to subtract the 1 from 10 and the other figures

from 9 respectively; this gives 9. But the best way to write it is to put a 0 for each of the places where 9 was subtracted from 9, thus 0009, and add it to 18931.

	18931
	0009
adding and throwing out 10000	8940

which is the remainder or answer.

There are three entries of cash due; \$109.50, \$186.71 and \$199.25; there is one entry of cash owing; \$118.33; calculate the balance by the use of the arithmetical complement.

	\$109.50	
	186.71	
	199.25	
a.c. of \$118.33	881.67	
adding and throwing out	\$377.13	the balance.

Subtraction of Sums

When the sum of several numbers is to be subtracted from the sum of several others, the usual way is to add each set of numbers separately and subtract the sum of one from that of the other set. But it can be done directly. In the example the three numbers below the upper line are to have their sum subtracted from the sum of those above the line. Proceed as follows: Add 2, 2 and 3, giving 7. Keep this in mind and add 9, 9 and 6, giving 24. Subtract 7 from 24 giving 17; put down the 7 under the lower line and carrying 1 add it to the next

```
 9876
 4969
 4989
 ----
 1223
 2112
 3212
 ----
13287
```

column above the upper line. This gives 22 from which is to be subtracted the sum of 2, 1 and 1; then 4 from $1 + 8 + 6 + 7$ gives 18; the 8 is put down and the 1 is carried to the next column above the line and the process is carried through to the end. If there is a figure in the ten place at the last, as there is in the example it is put down and the operation is complete. The next example shows a case where the carrying 1 process applies to the bottom figures, those of the subtrahend. First 9, 8 and 9 are added giving 26; keeping this in mind add 6, 0 and 2 giving 8. Then 6 (of the 26) is subtracted from 8 and the difference, 2, is put down below the bottom line, and 2 is carried but this time to the lower figures giving $2 + 8 + 6 + 7 = 23$. The figures in the upper column add up 16; 3 (of the 23) from 16 leaves 13; put down 3 and carry the difference between 2 (of the 23) and 1 (of the 16) to the next lower column. It is added to the lower column because the larger number, 2, belongs below the upper line. Following out the same method we next have to subtract 27 from 13; this gives 6 to be put on the lower row, and 2 to carry to the sum of the next lower column and 0 to be carried to the sum of the upper column; the net result is 2 to be carried to the next lower column. Then finally we have to subtract $2 + 2 + 1 + 2$ from the sum of $4 + 4 + 9$, which gives the last or left hand figures of the subtraction, 10, which are accordingly set down.

$$
\begin{array}{r}
9476 \\
4020 \\
4972 \\
\hline
2979 \\
1968 \\
2889 \\
\hline
10632
\end{array}
$$

The process is perfectly simple, the principal point being to keep correct on the carrying; you must carry to the proper place.

Complement Subtraction of Sums

The next method of doing the same kind of subtraction is a variation on the last. The figures in the lower colums are added and their sums are subtracted from the multiple of ten next above each one. Thus if they add up 14 in any column this is subtracted from 20, because that is the multiple of 10 next above 14. This gives 6, and the 6 is added to the upper column. Suppose the upper column with this addition of 6 adds up 23; then 3 is put down and as there were two tens in the number used to give the complement of the lower column and also the same number of tens in the sum of the upper column there is nothing to carry. Had the lower column required 20 for the subtraction giving the complement, and had the upper column added up 23, then the difference of the tens, i. e. one ten less two tens, would have to be subtracted from the next lower column. Had the tens in the upper column addition exceeded the tens used in getting the complement of the lower column the difference would have to be added to the next lower column.

```
 56243
 84164
  3452
 26348
──────
  2942
  3654
  2308
──────
161303
```

Referring now to the example, the first lower column adds up 14; its complement is obtained by subtracting 14 from 20 giving 6; this is added to the upper column and the sum is 23; 3 is put down and there is nothing to carry because the 2 of the 23 balances the 2 of the 20, the latter the number required in getting the complement of the lower column addition. The second lower column adds

up 9; its complement 10 — 9 is 1; this is added to the column above it giving 20; 0 is put down and the difference between 10 and 20 is made unitary by dropping the 0 and as the larger ten figure belonged to the upper column the difference, 1 is subtracted from the third lower column so that its sum becomes 17; its complement is 3, which is added into the upper column giving 13. The difference of complement number of the lower column, 20, and the tens in the sum of the upper column favors the lower column; the difference therefore is added to the fourth lower column, instead of being subtracted as before. This difference is 1; the addition gives 8 as the sum of the fourth lower column, its complement number is 10, and its complement is 2. This is added to the upper column and the sum is 21; 1 is put down and the difference of the tens, 1, is added to the fourth upper column, because there is no lower column to subtract it from. This gives 16, which is set down, and the operation is completed.

A Property of Subtraction

The following is in the line of properties of numbers.

Subtract one number from another; then subtract the second number from the first. In one case you will have to borrow ten, but disregard this and put down the unit figure of the difference. If single figures have been used the sum of the differences will be 10. If double figures have been used the differences added will give 100, and so on.

Some examples follow:

a.		*b.*		*c.*	
9	3	96	32	875	221
3	9	32	96	221	675
—	—	—	—	—	—
6	4	64	36	454	546

In example *a.* the remainders add up to 10, in the next example, *b.* the remainders add up to 100, and in the third to 1,000. It will be noticed that the "carrying one" process is left aside.

CHAPTER IV.

MULTIPLICATION

Multiplication a Short Method of Addition

Multiplication is a shortened method of doing addition. If we are to multiply 9 by 7, we at once go to the multiplication table, which everyone is supposed to know by heart, and put down 63. But if the operation is analyzed, it will be found that what has been done it this—seven 9's have been added to each other. Putting it in formula we have: $9+9+9+9+9+9+9=9\times7=63$. Addition of the seven 9's is the same thing in its result as multiplying 9 by 7.

The reverse holds; the addition of the nine 7's is the same in result as multiplying 7 by 9.

The Multiplication Table

The first and all-essential thing in multiplication is to know the multiplication table.

As almost universally taught the table includes as its upper limit, twelve times twelve.

At first sight the multiplication table seems much more formidable than it really is. We have seen that the addition table for two number addition is a very simple affair as the number of sums involved in it are concerned. It contains only 17 sums in 45 combinations.

The multiplication table up to twelve times seems to involve one hundred and forty four operations to be

memorized. On analysis it becomes much simpler. We may omit one times as not to be learned, because it is known to anyone who can count up to twelve. This leaves us one hundred and thirty two operations. Of these eleven are one times; such as 3 times 1 are 3, 4 times 1 are 4; leaving these out we have left one hundred and twenty one operations. But many of these are simple reversals of each other, such as 3×4 and 4×3. Counting two reversals as one operation, which is perfectly correct, the operations reduce to sixty seven and many products of these operations are repeated, so that the number of products is only forty nine. They are the following:

4 6 8 9 10 12 14 15 16 18 20 21 22 24 25
27 28 30 32 33 35 36 40 42 44 45 48 50 55
56 60 64 66 70 72 77 80 81 84 88 90 99 100
108 110 120 121 132 144

Extending the Multiplication Table

There is no particular reason why the multiplication table should stop at twelve times as it always does. Many teachers advocate its being continued up to twenty times or even twenty five times.

Without absolutely learning all the higher multiplications up to one or the other of these numbers, anyone, doing much calculating, will soon learn many of the multiplications. They can be made available too by taking advantage of the multiplication of the lower known products by 2 or by 4. Fourteen times is seven times multiplied by 2; sixteen times is eight times multiplied by 2 and so for other even multipliers. Then the squares of the higher numbers such as sixteen times sixteen may

be taken as four times sixty four, which last is the square of 8.

But when it comes to the prime numbers, thirteen, seventeen and so on, then there is no simple way, that is really practical, of obtaining them.

It follows that if the learning by heart of the multiplication table for the higher multiplications is to be accomplished, the odd number multiplications should be learned first.

Mutiplying by Double or Two Digit Numbers

It is always easier to multiply by a single number than by a double one. Suppose 29 is to be multiplied by 14. If twice 29 is multiplied by half of 14 the answer will be given. So we may proceed thus:

$$\begin{array}{r} 29 \times 2 = 58 \\ 14 \div 2 = 7 \\ \hline 406 \end{array}$$

There are any number of variations on this method; the general principle is to multiply or divide by a number which will make a single number out of one of the two given numbers. The trouble is sometimes that one of the numbers may be indivisible without a remainder.

Multiplication of Two Digit Numbers

Quantities of two digits and having the same figure in the tens place, such as 56 and 53, 69 and 64, can be multiplied as follows:

Multiply the units together; multiply the tens digit of one of the numbers by the sum of the units of the original numbers and annex an ought; multiply the tens figures

together and annex two oughts; add the three products for the final results—the product of the original numbers.

Multiply 63 by 69.

$3 \times 9 = 27$. This is the first of the three quantities. Multiply the tens figure, 6, by the sum of the units figures, $3 + 9 = 12$; the product is $6 \times 12 = 72$; annex an ought, giving 720, the second quantity. Multiply the tens figures together, giving 36, and annex two oughts for the third figure, 3600. The sum of the three is the product of 63 by 69. $27 + 720 + 3600 = 4347$.

Two other examples follow.

$$\begin{array}{rr} 97 \times 93 = 9021 & \qquad 91 \times 92 = 8372 \\ \hline 7 \times \ \ 3 = \quad 21 & \qquad 1 \times \ \ 2 = \qquad 2 \\ 9 \times 10 = \ \ 900 & \qquad 9 \times \ \ 3 = \ \ 270 \\ 9 \times \ \ 9 = 8100 & \qquad 9 \times \ \ 9 = 8100 \\ \hline 9021 & 8372 \end{array}$$

The ciphers or oughts have in the above operations been inserted directly, without indication of the operation.

Multiplication of Three Digit and Higher Numbers

This method can be extended to include much higher numbers if the computator knows the squares of large numbers. Two examples of the multiplication of larger numbers follow.

$$\begin{array}{rr} 259 \times 257 = 66563 & \qquad 303 \times 308 = 93324 \\ \hline 9 \times \ \ 7 = \qquad 63 & \qquad 3 \times \ \ 8 = \qquad 24 \\ 25 \times 16 = \ \ 4000 & \qquad 30 \times 11 = \ \ 3300 \\ 25 \times 25 = 62500 & \qquad 30 \times 30 = 90000 \\ \hline 66563 & 93324 \end{array}$$

As before the ciphers are inserted or annexed directly.

Increment Multiplication

Suppose there are two numbers of two figures each to be multiplied; by adding an increment to one and subtracting the same increment from the other, we make a decimal number out of one of the two. Then to the difference between the two original numbers add the increment and multiply this sum by the increment, and add it to the first product. The result will be the product of the two original numbers, if the increment was added to the larger and subtracted from the smaller number. But if the increment was added to the smaller and subtracted from the larger, then subtract the increment from the difference between the original numbers and multiply it by the increment, and subtract it from the product of the first multiplication. The first process is carried out in the left hand calculation—the second described one in the right hand one. In both the number 83 is to be multiplied by 62.

$$
\begin{array}{rr}
83+2=85 & 21 \\
62-2=60 & 2 \\
\hline
5100 & 23 \\
46 & 2 \\
\hline
5146 & 46
\end{array}
\qquad\qquad
\begin{array}{rr}
62+3=65 & 21-3=18 \\
83-3=80 & 3 \\
\hline
5200 & 18 \\
54 & 3 \\
\hline
5146 & 54
\end{array}
$$

In the left hand example the increment, 2, is added to the larger number and subtracted from the smaller; it is added to the difference between the two numbers, 21, and this sum is multiplied by the same increment, 2, and is added to the product of 85×60. In the left hand example the increment is 3, it is added to the smaller

number and subtracted from the larger. It is subtracted from the difference between the two numbers, 21, and the result, 18, is multiplied by the same increment, 3, and the product is subtracted from the product of 65 by 80.

The method is interesting but perhaps of little practical value.

Another Method of Increment Multiplication

The next method is a variation on the last one. To each number is added an increment such as will produce a decimal in each case. Then one of the original numbers is multiplied by the other one increased by its increment. This is a simple one figure multiplication. The first number increased by its increment is next multiplied by the increment of the second one, which is also a single figure multiplication; this product is subtracted from the first product. Then the product of the two increments is added to this and is the answer. It is not easy to put into words but it is really simplicity itself as will be seen in the examples.

Let the numbers to be multiplied be 687 by 893. The increments will be 13 for the first number and 7 for the second. The numbers increased by their respective increments will be 700 and 900. They may be arranged as follows:

$$687 + 13 = 700$$
$$893 + \ \ 7 = 900$$

Following the rule multiply 687 by 900; this gives 618300. Next multiply 700 by the increment of the other number, 7; this gives 4900. Subtracting we get 613400. To this we add the product of the increments, $7 \times 13 =$

91, which gives the answer, the product of 687 by 893; it is 613491.

We could have gone the other way and multiplied as below.

$893 \times 700 = 625100$. From this subtract the product, $900 \times 13 = 11700 = 613400$ as before. To this add $7 \times 13 = 91$, and we get the result, 613491.

The numbers to be multiplied may reduce to decimals more easily by subtraction of what are really decrements. Proceed exactly as before except that we have to add throughout. Take 805 and 512 as the numbers. The operation follows.

$$805 - 5 = 800$$
$$512 - 12 = 500$$

then $805 \times 500 = 402500$; $800 \times 12 = 9600$; and $5 \times 12 = 60$. Adding the three products gives the product of the original numbers, 412160.

Finally we may add to one number and subtract from the other. Take 812 and 395 as the two numbers to be multiplied. It is obvious that it is a case of adding and subtracting.

$$812 - 12 = 800$$
$$395 + 5 = 400$$

then $812 \times 400 = 324800$; $800 \times 5 = 4000$; $5 \times 12 = 60$; and the sum of the last two products subtracted from the first gives the answer, 320740.

Had we multiplied the other way, 395 by 800, and 400 by 12 and as before 5 by 12, it would have been necessary to add the first two products and to subtract the third. It is advisable therefore to either add increments to both

original numbers or to subtract decrements from both to avoid confusion.

Special Method of Multiplying Three Digit Numbers

The following is an interesting way of multiplying three figure numbers. Multiply 378 by 469. First multiply 378 by 60 this gives 22680. Now multiply 378 by 409 as follows: 9 times 378 are 3402. Write down only the figures, 02, carry the rest in your mind. Then start with 4 times 378. Four times 8 are 32; add this to the 34 and you have 66; write down 6 and carry the other 6; then four times 7 are 28 and 6 are 34; put down 4 and and carry 3; finally four times 3 are 12 and 3 to carry are 15. We now have 154602 to which is to be added 22680, giving 177282 the product asked for.

This method may not be very practical, but this way of multiplying by such numbers as 409, 307 and the like is excellent practice in mental arithmetic.

The method is a special case of cross-multiplication, which latter is given further on in this book.

Multiplication of Special Cases of Polynomials

In multiplication by polynomials it is often convenient to derive one product from another by multiplication or division, instead of recurring to the multiplication of the original multiplicand.

Suppose a number is to be multiplied by 63. After it has been multiplied by 3 in the regular order, instead of multiplying it by 6, the product already obtained may be multiplied by 2, which will give the identical result, as if the original number had been multiplied by 6, because the successive multiplications by 3 and by 2 are the same in effect as multiplying by 6.

We will now set down the multiplication of a quantity by 63.

$$\begin{array}{r} 3982 \\ 63 \\ \hline 11946 \\ 23892 \\ \hline 250866 \end{array}$$

The first product is the result of multiplying the multiplicand by 3; the second product may be obtained by multiplication of the same quantity by 6. But by the method under discussion, it is obtained by multiplying the first product, 11946 by 2.

The operation as written down does not show whether the work is done by the one or by the other method.

Suppose the multiplier had been 36. Then the first product would have been 23892, and the second product would have been obtained by dividing this product by 2, and putting it down one place to the left, just as in regular multiplication. The products and their sum would be

$$\begin{array}{r} 23892 \\ 11946 \\ \hline 143352 \end{array}$$

This method may be applied when numbers divisible by one another are scattered about among other numbers. Thus suppose a quantity was to be multiplied by 27633. First multiply by 3; copy this down one place to the left for the seocnd product; multiply this product by 2 for

the third product; multiply the original multiplicand by 7 for the fourth product; and finally for the last product divide the third product by 3. An example follows:

```
    2971
   27633
   -----
    8913
   8913
  17826
 20797
 5942
--------
82097643
```

This shows nothing out of the ordinary way, but it will be seen by inspection how it can be done by the processes detailed. Another thing to be observed is that if the multiplier and multiplicand changed places the process could not be applied. Thus set the quantities down thus:

```
27633
 2971
-----
```

In this case multiplication can only be done by the regular method, if we take the lower quantity for the multiplier. Of course there is nothing to prevent us from multiplying upwards, using the upper quantity as the multiplier; then the method in question is again applicable.

The method lends itself to many variations; it is an excellent way of proving the correctness of multiplications done by the regular method.

Inverted or Left Hand Multiplication

Inverted multiplication is the term sometimes applied to the following method, where the left hand figures are first multiplied and then the next and the whole then added together.

To multiply 79 by 9 first multiply 70 by 9, which gives 630; then multiply the 9 of the 79 by 9 which gives 81; adding, $630 + 81 = 711$.

Now take a three number quantity; multiply 634 by 8. We have:

$$\begin{array}{r} 600 \times 8 = 4800 \\ 30 \times 8 = 240 \\ 4 \times 8 = 32 \\ \hline 5072 \end{array}$$

which is the answer.

Factorial or Proportional Multiplication

Factorial or proportional multiplication is when one of the numbers is multiplied and the other is divided by a quantity which will simplify the operation. Suppose we want to multiply 29 by 14. We can simplify matters by dividing 14 by 2. To carry out the proportion we shall then multiply 29 by the same quantity, 2. This gives $29 \times 14 = 58 \times 7 = 406$.

Take $22\frac{1}{2} \times 36$. We take 4 as the quantity to multiply and divide by. Then $22\frac{1}{2} \times 36 = 90 \times 9 = 810$.

In the last example $22\frac{1}{2}$ was multiplied by 4 and 36 was divided by it. If either of the numbers can be simplified by division we are always free to multiply the other by the same quantity. The reverse is not always the case. Multiplication may simplify one of the numbers where the other is quite intractable to division by the

multiplying number. If 45 were to be multiplied by 27, multiplication of 45 by 2 would give 90, a very simple number to multiply by, but we cannot divide 27 by the same 2. But this difficulty may be met by the following method.

If multiplication will give a simpler multiplier, multiply the multiplier, and then do the final multiplication, and divide the product by the number used to simplify one of the quantities. Let us take 431×45.

Multiply 45 by 2; this gives 90, an easy number to multiply by. $431 \times 90 = 38790$. Divide this by 2 and we have 19395, which is the product of the two original numbers, $431 \times 45 = 19395$.

Under this process fall a quantity of numbers; namely those ending in 5 up to 55 give multipliers within the range of the multiplication table.

Multiply 269 by 55. Doubling 55 gives 110. $269 \times 110 = 29{,}590$, and half of this is 14,795.

As an alternative in this case we would equally as well have multiplied by 11 and by 5 in succession.

Aliquot Part Multiplication

An aliquot part of a number is a number which can be a divisor of the number without any remainder. It is a factor. Thus 5 is an aliquot part of 15 or of 25, because either of the numbers in question can be divided by 5 without any remainder; it is a factor of either number.

Aliquot parts of 100 are often of much use in abbreviating operations; suppose we want to know the sixteenth part of 2000; from aliquot parts we know that the sixteenth of 100 is $6\frac{1}{4}$; therefore the sixteenth part of 2000

is 125, for that is the product of $6\frac{1}{4}$ by 20, and $100 \times 20 = 2000$.

In the statement of aliquot parts we simply give the equivalents of aliquot parts of 100 as fractions. Each aliquot part is given as a fraction of 100.

2......$\frac{1}{50}$	$12\frac{1}{2}$......$\frac{1}{8}$	60......$\frac{3}{5}$
4......$\frac{1}{25}$	$13\frac{1}{3}$.....$\frac{2}{15}$	$66\frac{2}{3}$.....$\frac{2}{3}$
5......$\frac{1}{20}$	$16\frac{2}{3}$......$\frac{1}{6}$	75......$\frac{3}{4}$
	20.......$\frac{1}{5}$	
$6\frac{1}{4}$.....$\frac{1}{16}$	25.......$\frac{1}{4}$	80......$\frac{4}{5}$
$6\frac{2}{3}$.....$\frac{1}{15}$	$33\frac{1}{3}$......$\frac{1}{3}$	$83\frac{1}{3}$.....$\frac{5}{6}$
$8\frac{1}{3}$.....$\frac{1}{12}$	50.......$\frac{1}{2}$	$87\frac{1}{2}$.....$\frac{7}{8}$

As aliquot parts refer to 100, they are really hundredths, and the left hand columns with a cipher preceding them, can be written as decimals—.02, .04 &c. until we get to the double figure numbers; these with the decimal point read at once as hundredths, thousandths &c. without any cipher preceding them—.12½ or more properly .125, .1333..., &c.

As these aliquot parts refer to 100 parts as their basis, they are useful in calculations involving dollars; the third of a dollar is 33⅓ cts. the fifth of a dollar is 20 cts.

The above set of parts is far from complete, for there are any number of aliquot parts, which may be of use, with various bases. Once the idea of how to use them is grasped their use may be greatly extended according to the operations to be done.

Aliquot parts then are of constant use in multiplications. The following examples will show how they can be used.

To multiply by 50 annex two ciphers and divide by 2. $32 \times 50 = 3200 \div 2 = 1600$.

To multiply by 25 annex two ciphers and divide by 4. $28 \times 25 = 2800 \div 4 = 700$.

To multiply by 20 annex two ciphers as hitherto and divide by 5. In this case it would generally be easier to multiply directly, but there may be reasons for doing it in the indirect way.

The operations may be combined. Thus to multiply by 75, annex two ciphers and divide first by 4; then divide the original number with the two ciphers by 2 and add the quotients. $29 \times 75 = 2900 \div 4 = 725$ and $2900 \div 2 = 1450$ and $725 + 1450 = 2175$.

Practical Use of Aliquot Parts in Multiplication

One of the conveniences of the use of aliquot parts is that they enable us to dispense with fractions. They are only applicable, as given here, to hundreds and other decimal numbers, but as our coinage is decimal they have extensive application in business calculations. If the metric system is being used, then again they may be useful.

Some examples of their applications follow.

What is the cost of 55 articles at \$2.50 each? Add a cipher and divide by 4; $^{550}/_{4} = \$137.50$.

Multiply 63 items by \$2.12½. Multiply 63 by 2, and add $^{63}/_{8}$, which is 7⅞ or \$7.87½; this gives \$133.87½.

27 yards at 62½ cts. is to be charged. This is ⅝ of \$1.00; as $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$ we add together $^{27}/_{2}$ and $^{27}/_{8}$, which gives $13.50 + 3.37\frac{1}{2} = \$16.87\frac{1}{2}$.

Multiply 28791 by 125. This may be done in either of two ways. We may add three ciphers and divide by 8; this is because the list of aliquot parts tells that 125 is ⅛ of 1000; this gives 3598875. Or we may multiply

by 100 and add one quarter of the product thereto, because 25 is one quarter of 100; $2879100 + 719775 = 3598875$ as before.

Suppose it is asked what the product of 1000 by $5/8$ is. This is given directly by aliquot parts; it is 625. Or to 500, which is $4/8$ of 1000, may be added $^{100}/_8$, or $1/8$ of 1000, this is 125, and adding the two gives 625 as before.

There is another thing to be said about aliquot parts, and this applies to many fractional calculations, where a mixed number occurs. It may be told by examples.

Multiply 100 by $2\frac{1}{4}$. This is done in one way by multiplying the multiplicand, 100, by 2 and then by $1/4$ and adding the two for the answer. Thus we have $100 \times 2 = 200$; $100 \times 1/4 = 25$; $200 + 25 = 225$. This is simple but there is another way of doing it, which may be more convenient in some cases. The original number may be multiplied by the integral number of the multiplier, and then the product may be multiplied by one half the fraction and the two are then to be added together. In the case just used, 100 would be multiplied by 2, giving 200, and to that would be added $^{20}/_8$, which is 25, and the result will be 225, the same as before.

If the multiplier had been $4\frac{1}{2}$, then to the first product, 400, there should have been added $1/16$ of 400. The result is the same of course in all methods.

This method is of assistance in some fraction work. Suppose a number is to be multiplied by $2\frac{2}{3}$. It is manifestly easier to multiply the number by 2 and add to the product $1/3$ of the product than it is to add to the same product $2/3$ of the original number. For to multiply by $1/3$, we have only to divide by 3; to multiply by $2/3$ we have to multiply by 2 as well as to divide by 3; one is twice the work of the other.

Mixed numbers, not exactly within the scope of this treatment, may often be brought within it. Suppose it is a question of multiplying some number by $6\frac{3}{4}$; this mixed number can be written $6\frac{6}{8}$, for $\frac{6}{8} = \frac{3}{4}$. Then to multiply by it, the multiplicand is multiplied by 6 and there is added to the product $\frac{1}{8}$ of the product. It is evident that $\frac{1}{8}$ of the product is equal to $\frac{6}{8}$ of the multiplicand. It is easier to do this than to proceed in the regular way and multiply the original number, the multiplicand, by 3 and divide the result by 4 and thus obtain the quantity to be added to the first product.

Multiply 77 by $8\frac{2}{5} = 8\frac{8}{20}$. Multiplying 77 by 8 gives 616. To this add $616 \times \frac{1}{20}$, or what is the same thing, divide 616 by 20 and add. This gives $616 + 30.80 = 646.80$. In the regular way we should have multiplied 77 by 2 and divided by 5 to get the quantity to be added to 615. The first way is the easier.

Aliquot parts, as given here, refer to 100 as the base. The principle involved in their use may be extended by factor multiplying. This method is of such extensive and of so many special applications, that it can best be given by means of examples and by the statements of special cases. The computator will have no difficulty in applying it to a new case, once the idea is grasped.

Factor Multiplying

To do factor multiplying one of the numbers to be multiplied is factored; this one is taken as the multiplier and the other one is multiplied by the factors in succession.

To multiply by any even number between 12 and under 25. Divide the multiplier by 2; multiply the other number by this quotient and then multiply this product by 2.

Multiply 1986 by 18. 18 divided by 2 gives 9. $1986 \times 9 = 17{,}874$; $17{,}874 \times 2 = 35{,}748$, the answer. The first multiplication could have been by 2 and the second by 9.

A number of multipliers, up to 36 in value can be brought within the range of the multiplication table by division by 3.

Multiply 3986 by 36. In accordance with the principle we divide 36 by 3; this gives the two factors 3 and 12; successive multiplication by these gives the product; $3986 \times 12 = 47{,}832$ and $47{,}832 \times 3 = 143{,}496$.

Division by 4 carries the principle up to 48 for numbers divisible by 4. In this way we can go as high as 108 for numbers divisible by 9. By using three successive multiplications by three factors of the multiplier the method can be greatly extended; still greater numbers of factors can be used until the limit of usefulness is passed.

Thus it is as easy to multiply by 243 directly, as it is by 3, by 9 and by 9 again in succession. These are three factors of 243 which could be used for factor multiplying.

This method applies well to numbers ending in one cipher or in several ciphers. To multiply by 180, use as successive multipliers 2 and 90, or 3 and 60.

We have seen that to multiply by 25 we may add two ciphers and divide by 4. Suppose a number is to be multiplied by 24; it is clear that the two ciphers may be added and the division by 4 will give the product one time too high. Therefore all that is necessary is to multiply by 25; this is done by adding two ciphers and dividing by 4; then subtract the number. Suppose 243 is to be multiplied by 124. First multiply by 100; this takes care of the initial 1. Then add two ciphers, divide by 4 and add

it to the product just obtained. Finally subtract 243 and you will have the correct result. The operation follows:

$$
\begin{array}{lr}
243 \times 100 = & 24300 \\
24300 \div 4 \quad = & 6075 \\
\hline
 & 30375 \\
\text{subtract } 243 & 243 \\
\hline
 & 30132
\end{array}
$$

If 26 were the multiplier, the multiplicand would be added instead of subtracted.

To Multiply by Nine

To multiply a number by 9, write the number down and imagine a cipher placed after it. Then subtract the number from the unwritten one. It amounts to subtracting the right hand figure of the original number from 0, carrying one and subtracting the next figure of the original one increased by the 1 carried from the first figure and so on.

$$
\begin{array}{r}
67583 \\
\hline
608247
\end{array}
$$

Thus in the example we say 3 from 0 leaves 7 and 1 to carry; adding this 1 to the second figure of the original, 8, gives 9, and 9 from 3 gives 4 with 1 to carry. Carrying the one to the next figure, 5, we have 6 from 8 leaves 2 with nothing to carry. 7 from 5 gives 8 with 1 to carry; carrying the 1, 7 from 7 leaves 0 with nothing to carry, so finally the 6 is put down completing the multiplication. Had there been 1 to carry at the last the 6 would have been 5 in the product at the left hand.

To Multiply by Eleven

To multiply by 11 write down the number. Then start by putting down its right hand figure as the first figure of the product. Then add the second figure to the first and put down the right hand figure of the sum if there are two figures in it and if there are carry 1. If there is only one figure put that down, there being no 1 to carry. Add the third figure to the second adding 1 if it is to be carried and write the first figure down as before. In the example 3 is put down; then $8 + 3$ gives 11; 1 is put down and 1 is carried. Then $1 + 5$ are 6 and $6 + 8$ are 14; the 4 is put down and 1 is carried. $1 + 7$ are 8 and $8 + 5$ are 13; 3 is put down and 1 is carried. $1 + 6$ are 7 and $7 + 7$ are 14; 4 is put down with 1 to carry. This is added to the final figure, 6, and 7 is written down as the right hand figure of the product.

$$\begin{array}{r} 67583 \\ \hline 743413 \end{array}$$

To Multiply by One Hundred and Eleven

To multiply by 111 the operation is the same essentially as the one last described. To multiply the same number by 111 first put down the 3 for the right hand number; then add 8 and 3 giving 11 and put down 1; then add the 1 to be carried to $5 + 8 + 3$ and put down 7 and carry one; then $1 + 7 + 5 + 8$ put down 1 and carry 2, and so on.

Complement Multiplication

Complement multiplication may be done in a simple way by taking into the operation the true complements of the numbers, and may be made to include numbers varying in the ten places.

Subtracting each number from 10, 100 or other power of ten gives its complement. Multiply the complements together and add to the product the difference between either number and the complement of the other multiplied by 100.

Two examples are given, one of numbers with the same ten figures, the other with figures having different ten figures.

Multiply 97 by 93.

Complements are 3 and 7.

$3 \times 7 =$	21
$97 - 7$ or $93 - 3 = 90$	
$90 \times 100 =$	9000
	9021

Multiply 87 by 95.

Complements are 13 and 5.

$13 \times 5 =$	65
$87 - 5$ or $95 - 13 = 82$	
$82 \times 100 =$	8200
	8265

It makes no difference which number is diminished by the complement of the other as the result is the same.

The next example shows the same operation applied to three digit numbers. Here the complement is the difference between the number and 1000, and the difference of the complement of one of the numbers and the other number has to be multiplied by 1000 instead of by 100.

Multiply 931 by 972.

The complements are 69 and 28.

$931 - 28$ or $972 - 69 = 903$.

$$\begin{array}{r} 69 \times 28 = 1932 \\ 903 \times 1000 = 903000 \\ \hline 904932 \end{array}$$

Multiplication of Numbers Ending with 5

To multiply two numbers together, each of two digits and both ending in 5, proceed as follows: Put down 25 in the right hand place; multiply together the figures to the left of the fives and put the product next to the 25 on the left; then add half the sum of the figures to the left of the fives, taking them in their proper decimal place,

Multiply 45 by 25. Put down 25 to the right; multiply 4 by 2 giving 8; to this add half the sum of 4 and 2, which added to the 8 already found gives 11, which is put to the left of the 25; the result is 1125.

Multiply 35 by 45. Here the sum of the tens is uneven. Proceeding as before put down 25; put in front of it the product of the tens and add half their sum; the result is 1575. The operation is done as shown here:

$$\begin{array}{r} 5 \times 5 = \quad 25 \\ 30 \times 40 = 1200 \\ 30 + 40 = 70;\ 70/2 = \quad 35 \\ \hline 1575 \end{array}$$

Multiplying by Two Numbers at Once

To multiply by two numbers at once multiply the two figures of the multiplier by each figure of the multiplicand, writing down the last figure and carrying the one or more left hand figures. There is no object in doing it, if the whole operation has to be done at length. In the example the work placed on the right should be done mentally in practice.

1575	$5 \times 23 = 115$
23	$7 \times 23 = 161$
———	$5 \times 23 = 115$
36225	$1 \times 23 = 23$
	———
	36225

Next as a development of the process we will put down the four products obtained, with the numbers to be carried added in; then taking the last figures of the set and putting the last left hand figure obtained in front we again get the result. If the method is applied these four products will be: 115, 172, 132, 36, and the last digits in reverse order, with the 3 of the 36 last obtained in front, gives the answer or product, as before.

Multiplying by Any Number between Twelve and Twenty

To multiply by any number between 12 and 20, the following method is of interest.

Multiply the figures of the multiplicand in the regular order one by one by the unit figure of the multiplier. If there is anything to carry add it as usual, and add also to each product thus obtained the figure of the multiplicand on the right of the figure multiplied. Put down the right hand figure of the number thus obtained and carry the left hand figure if there is one. The tens figure of the multiplier does not enter into the calculation directly; it is taken care of in the process as described.

An example follows, with the successive steps in detail.

$$
\begin{array}{r}
39712 \\
17 \\
\hline
675104
\end{array}
$$

$7 \times 2 = 14$. Put down 4 and carry 1. *b.* (7×1)
$+ 1 + 2 = 10$. Put down 0 and carry 1. *c.* (7×7)
$+ 1 + 1 = 51$. Put down 1 and carry 5. *d.* (7×9)
$+ 5 + 7 = 75$. Put down 5 and carry 7. *e.* (7×3)
$+ 7 + 9 = 37$. Put down 7 and carrying 3 add it to the left hand figure of the multiplicand and put down the sum as the last figure.

In carrying out this method it is important to attend to the addition of the left hand figure to the last figure carried.

"Multiplying by the 'Teens"

This operation, when the multiplier is below 20 in value and exceeds 10 is called "multiplying by the teens." It is a modification of cross-multiplication, to the extent that cross-multiplication includes the possibility of any sized multiplier. Its use is only limited by the ability of the computator. Cross multiplication is an excellent test object to determine a computator's powers. If the multiplier is a large number it is very difficult to cross-multiply successfully.

Cross-Multiplication

Cross-multiplying is a method of multiplying by a number or quantity of more than one digit, without putting down the partial products. It will be best explained by examples. The following principles apply to its execution.

Call the right hand digit of any quantity the first number of the quantity; the next digit to the left the second and so on. Then if a quantity is multiplied by another, any single number or digit of the multiplicand multiplied by the first digit of the multiplier, will have

the first figure of this product directly under itself. If multiplied by the second digit the first figure of the product will be shifted one place to the left; if multiplied by the third digit it will have its first figure shifted two places to the left. If the product of a digit contains two figures, the second one will fall into its regular place to the left of the first one. We will now proceed to give some examples.

Multiply 72 by 63. Call 63 the multiplier.

Then: $2 \times 3 = 6$, the first figure of the product of the two original numbers. Next: $2 \times 6 = 12$ added to 7×3 gives $12 + 21 = 33$ and 3 is the second number of the product. We have therefore 3 to carry. Finally $7 \times 6 = 42$ and carrying the 3 gives 45, the third and fourth figures of the product, which is 4536.

Multiply 81 by 37, calling 37 the multiplier. It is done in formulas.

81×37. $7 \times 1 = 7$; the first number of the product.
$(7 \times 8) + (3 \times 1) = 59$; 9 is the second number with 5 to carry.
$(8 \times 3) + 5 = 29$; the third and fourth figures of the product.

The entire product is 2997.

Multiply 736 by 84; the operations are in formulas.

736×84. $6 \times 4 = 24$; 4 is the first figure; 2 to carry.
$(6 \times 8) + (3 \times 4) + 2 = 62$; 2 the second figure; 6 to carry.
$(3 \times 8) + (7 \times 4) + 6 = 58$; 8 the third figure; 5 to carry.
$(7 \times 8) + 5 = 61$; the fourth and fifth figures.

The entire product is 61824.

It will be observed that every figure in the multiplicand is multiplied by every figure in the multiplier, and that each digit of the products falls into its proper place.

Multiply 429 by 643; the operations are in formulas.

429×643. $3 \times 9 = 27$; 7 the first figure; 2 to carry.

$(9 \times 4) + (2 \times 3) + 7 = 44$; 4 the second figure; 4 to carry.

$(9 \times 6) + (2 \times 4) + (4 \times 3) + 4 = 78$; 8 the third figure; 7 to carry.

$(2 \times 6) + (4 \times 4) + 7 = 35$; 5 the fourth figure; 3 to carry.

$(4 \times 6) + 3 = 27$; the fifth and sixth figures.

The entire product is 275,847.

Multiply 3987 by 4926. The operations are in formulas.

3987×4926. $7 \times 6 = 42$; 2 the first figure; 4 to carry.

$(7 \times 2) + (8 \times 6) + 2 = 66$; 6 the second figure; 6 to carry.

$(7 \times 9) + (8 \times 2) + (9 \times 6) + 6 = 139$; 9 the third figure; 13 to carry.

$(7 \times 4) + (8 \times 9) + (9 \times 2) + (3 \times 6) + 13 = 149$; 9 the fourth figure; 14 to carry.

$(8 \times 4) + (9 \times 9) + (3 \times 2) + 14 = 133$; 3 the fifth figure; 13 to carry.

$(9 \times 4) + (3 \times 9) + 13 = 76$; 6 the sixth figure; 7 to carry.

$(3 \times 4) + 7 = 19$; the seventh and eighth figure.

The entire product is 19,639,962.

Cross-multiplication is the *ne plus ultra* of this operation; if you can do it with a four or five digit multiplier and an equally long or longer multiplicand, you may consider yourself a proficient. It has been explained by examples; after these are gone through carefully, the principle will be understood. It is hard to give a verbal rule of any value for the process.

In using it the operations should not be put down; nothing but the result should appear, figure by figure. By practice great expertness can be attained. Some can do it with numbers of twelve or more digits.

Slide Multiplication

Cross-multiplication can be done in another way called the sliding method. The multiplier is written out on a separate slip of paper but in inverted or reversed order. It is placed over or under the multiplicand and is shifted constantly one place to the left in succession after the multiplications indicated by its positions are done. We will repeat some of the examples already done by the regular method of cross-multiplication, doing them by the sliding method.

Multiply 72 by 63. Write the multiplier reversed upon a slip of paper, thus: 36. Place it over the multiplicand with its left-hand figure over the right-hand figure of the multiplicand, and multiply as indicated.

```
  36
72   3 × 2 = 6; the first figure of the product.
```

Now slide the paper slip one place towards the left; two multiplications will now be indicated.

36
72 $(6 \times 2) + (3 \times 7) = 33$; put down 3, the second figure; 3 is to carry.

Next slide the paper slip one more place to the left and multiply as indicated.

36
72 $(6 \times 7) + 3 = 45$; the third and fourth figures of the product.

The entire product is 4,536.

Multiply 429 by 643. Write 346 on the slip of paper, and proceed as in the last example.

346
429 $3 \times 9 = 27$; 7 the first figure; 2 to carry.

The slip of paper is shifted one place to the left.

346
429 $(9 \times 4) + (2 \times 3) + 2 = 44$; 4 the second figure 4 to carry.

The paper is shifted one more place to the left.

346
429 $(9 \times 6) + (2 \times 4) + (4 \times 3) + 4 = 78$; 8 the third figure; 7 to carry.

The paper is shifted one more place to the left.

346
429 $(2 \times 6) + (4 \times 4) + 7 = 35$; 5 the fourth figure; 3 to carry.

The paper is shifted to the left for the last time.

346
429 $(4 \times 6) + 3 = 27$; the fifth and sixth figures.

The entire product is 275,847.

After sufficient practice the paper slip can be dispensed with; the multiplier can be written in reverse order over or under the multiplicand and the work can be performed mentally, so that nothing is done on paper except the writing down the answer, digit by digit. There is not the least difficulty in doing it, but it is well to practice with the slip of paper first.

Excess of Nines Proof of Multiplication

The excess of nines method of testing the accuracy of multiplication is of special interest as an application of one of the properties of the number nine. If the digits of a number are added together and their sum is divided by 9 the remainder if there is any, is called the excess of nines. Instead of adding all the digits the best way of getting the excess is to do as in the example.

Required the excess of nines in 192846. Proceed as follows: 9 and 1 are 10, an excess of 1. Add this to the next digits; 1 and 2 and 8 are 11, an excess of 2; then as before $2+4+6=12$, an excess of 3. The quantity, 3, is reached by subtracting 9 from the last sum, 12, or it may be found by adding the digits of 12 together, $1+2=3$.

If a multiplication of two quantities is correctly performed, the excess of nines in the multiplicand multiplied by the excess in the multiplier and the excess in this product taken will equal the excess of nines in the product of the multiplication.

Test the accuracy of the following multiplication by excess of nines: $39821 \times 8769 = 349{,}190{,}349$.

The excess of nines in the first number, the multiplicand, is 5. The excess in the second number, the multi-

plier, is 3; the product is 15, and the excess of nines in 15 is 6. The excess of nines in the product of the original multiplication is also 6, so the operation comes out right by the test. It is to be remembered that this is only a test, not an absolute proof. If the excess comes out differently then the operation is certainly wrong.

Curiosities of the Multiplication Table

There are a number of curious things about the multiplication table, which things are of more interest than utility. Some of them will be given here.

If the products of any of the divisions of the table, such as three times or four times, are written down, and if their unit figures are added up, the sum of the figures in question, stopping at the ninth place, will be either 40 or 45 except in the case of five times. For even number multiplications such as four times or six times the sum of these units will be 40; for the odd times, three times or seven times, the sum will be 45. A number of examples are given below:

Three times:	Four times:	Six times:	Seven times:
3	4	6	7
6	8	12	14
9	12	18	21
12	16	24	28
15	20	30	35
18	24	36	42
21	28	42	49
24	32	48	56
27	36	54	63
—	—	—	—
45	40	40	45

The left-hand columns are not added up; the right-hand columns are the only ones added. The five times section is not put down; this gives as the sum of its left-hand digits only 25. If now the sum of the odd numbers in any uneven number section, such as three times or seven times are added, their sum will be 25. It is done here for three times, seven times and nine times:

Three times		Seven times		Nine times	
	3		7		9
	9		21		27
	15		35		45
	21		49		63
	27		63		81
	—		—		—
	25		25		25

It is, as before, only the right-hand numbers which have been added up, and their sum is the same as the sum of all the right-hand digits of the five times division of the multiplication table.

Now take any of the columns and add, this time horizontally, the component digits of the different products; taking the three times column given above, its digits add up thus: 3, 6, 9, 3, 6, 9, 3, 6, 9. The next column adds up thus: 4, 8, 3, 7, 2, 6, 10, 5, 9. The next column, six times, gives: 6, 3, 9, 6, 3, 9, 6, 12, 9, and if the digits of the anomalous looking 12 are added together they give the missing 3.

Various degrees of regularity can be traced out for these additions; the tracing and following them out may be left to the reader. Doing this constitutes a sort of "Arithmetic Solitaire."

Write down the nine digits in a vertical column, be-

ginning with 1 and ending with 9. Then to the right of this column write another one made up of the same nine digits, beginning this time with 9, one space above the 1 of the other column; by the side of the 1 of the first column place the 8 of the new column and so until all nine figures are written out. Then the full nine times of the multiplication table will be seen from nine times one up to nine times nine. The two columns are given at the side of the page.

9
18
27
36
45
54
63
72
81

A Curious Method of Multiplying

It is possible to multiply any numbers together and use only simple addition, multiplication by 2 and division by 2.

Put the two numbers down side by side. Divide one of them by 2, put the quotient under the same number and divide this quotient by 2. Pay no attention to remainders. Repeat this until you can go no further, or until the quotient 1 is obtained. Multiply the other number by 2, put it alongside of the first quotient, multiply this by 2 and put it alongside of the second quotient. Keep this up until you have a multiple for each of the quotients. The quotients can go only a definite distance and are the limiting element. Of the products thus obtained strike out each one that is opposite an even number quotient; the sum of the products remaining will give the product of the two original numbers.

Multiply 68 by 91.

It is immaterial which number is successively divided and which multiplied. It is done in both ways here, indicated as *a* and *b*.

a	*b*	
6891	68	91
34182	136	45
17364	272	22
8728	544	11
41456	1088	5
22912	2176	2
15824	4352	1

The quantities opposite the odd number quotients have to be added to give the product of the two original numbers. This is done below in each case below its own calculation.

364	68
5824	136
	544
6188	1088
	4352
	6188

Oddities in Multiplication

If the digits with the omission of 1 are written in reverse order and multiplied by 9, the product will be a succession of nine 8's.

$$98765432 \times 9 = 888888888$$

Now write the nine digits including the 1 this time, and multiply by 9 and the product will be the nine 8's as before with a 9 on the right end. If multiplied by 18, the left hand figure of the product will be a 1, then will come nine 7's and as the right hand figure there will be an 8.

If multiplied by 27, the left hand figure will be a 2, the nine middle figures will be 6's, and the right hand figure a 7. It goes thus through the multiples of 9 used as multipliers, until we finally get for nine times nine or 81 as a multiplier, the left hand figure, 8, the right hand figure, 1 and ciphers to the regular number of nine, as the intermediate digits.

In each product the left and right hand figures give or repeat the multiplier, and the intermediate figures run in regular order from 8's to ciphers. Some of the multiplications are given here:

987654321 × 9 = 8888888889
ditto × 18 = 17777777778
ditto × 27 = 26666666667
* * * * * * *
* * * * * * *
ditto × 81 = 80000000001

Of course for the left hand figure of the product of the multiplication by 9, there is no left hand figure to be supplied; the final or right hand 9 gives the multiplier.

The number 15873 multiplied by 7 gives as product six 1's.

$$15873 \times 7 = 111111.$$

Now multiply this same number by 9 and the product is 142857, and this number multiplied by 7 gives as product six 9's.

$$15873 \times 9 = 142857 \quad \text{and} \quad 142857 \times 7 = 999999$$

or

$$15873 \times 63 = 999999$$

The following is an odd series of products. As usual it is the number 9 that is the main factor in the operations.

$$9 \times 9 = 81 \text{ and } 81 + 7 = 88$$
$$9 \times 98 = 882 \text{ and } 882 + 6 = 888$$
$$9 \times 987 = 8883 \text{ and } 8883 + 5 = 8888$$

The last two in the series are:

$$9 \times 9876543 = 88888887 \text{ and } 88888887 + 1 = 88888888$$
$$9 \times 98765432 = 88888888 \text{ and } 88888888 + 0 = 88888888$$

The number 153846 multiplied by 13 produces 1 as the left hand digit, 8 as the right hand digit, with five 9's between them.

$$153846 \times 13 = 1999998$$

In many cases these curious multiplications can be carried out by further multiplications, so as to give other results. We have seen that if we multiply 153846 by 13 the product is 1999998. If we add one half of itself to the above number, namely 76923 we obtain 230769, and this multiplied by 13 gives 2999997. Adding it again to the last sum we obtain 307692, and multiplying this by 13 we get 3999996. This process can be carried down to the eighth product, which will be 153846 multiplied by 5, for that is what the successive additions of 76923 leads to; the product of 153846 by 5 is 769230 and the product of this number by 13 gives 9999990. Now these products may be placed in columns; the first three are given in full.

$153846 \times 13 = 1999998$	5999994
$230769 \times 13 = 2999997$	6999993
$307692 \times 13 = 3999996$	7999992
$384615 \times 13 = 4999995$	8999991

The left hand digits run from 1 to 8; the right hand ones run from 8 to 1; the left and right hand digits give the products of 9 by 2, 3 &c.

Other odd multiplications are the following:

$$37037037037 \times 9 = 333333333333$$
$$13717421 \times 9 = 123456789$$
$$987654321 \times 9 = 8888888889$$

Finger Multiplication

The following way of multiplying two numbers each more than 5 and less than 10, is more curious than useful, for everyone is supposed to know the multiplication table.

Suppose we are to multiply 9 by 8. Hold up the fingers of both hands open. Take the difference between one of the numbers and 10 and bend down a finger or fingers corresponding in number to that difference. Do this on one hand for one of the numbers and on the other hand for the other number. The fingers not bent down give the tens and the product of the fingers bent down on one hand by those on the other hand give the units. In the case of 9 and 8 two fingers will be bent down on one hand and one on the other; their product is 2; there are seven fingers left standing; these are the tens; putting them before the 2 we have 72, the answer.

This operation can be carried out very well on an abacus.

If the fingers to be held down exceed the number 10, subtract the excess from the tens of the product.

CHAPTER V

DIVISION

Factor Division

If a number is to be divided by one so large that it requires long division, if the divisor can be factored down far enough a series of short divisions by the factors can be substituted for the long division.

In the example 30672 is divided by 432. The latter can be factored thus: $12 \times 12 \times 3$, and three short divisions by these factors give the quotient, 71.

$$\begin{array}{r} 12\,\underline{)\,30672} \\ 12\,\underline{)\,2556} \\ 3\,\underline{)\,213} \\ 71 \end{array}$$

This division has no remainder. Let it be required to divide 34577 by 18; here there is a remainder. Divide by the factors, 2, 3, and 3. Here there are three remainders; each remainder is to be multiplied by the factors of the divisions preceding its own and the sum of the products is the total remainder. The first remainder is not multiplied by anything; this is under the rule for no division precedes it.

$$\begin{array}{rl} 2\,\underline{)\,34577} & \\ 3\,\underline{)\,17288} & 1\ldots.1 \\ 3\,\underline{)\,5762} & 2\ldots.5 \\ 1920 & 2\ldots17 \end{array}$$

The remainder from the division of the whole number, 34577 by the first factor, 2, is 1. The remainder from the next division, which is the division of half the number by 3 is 2. This is multiplied by the first divisor, 2, and is added to the first remainder, 1, giving 5. The next

remainder comes from the division of $\frac{1}{6}$th the number by 3. This remainder is multiplied therefore by 3 and by 2, or by 6, and is added to 5 just determined, giving 17 which is the total remainder.

Or start at the bottom and multiply the last remainder, by 3 and add it to the next remainder above it; this gives $(2 \times 3) + 2 = 8$. Multiply 8 by 2 and add the first remainder, 1, and the total remainder is obtained, namely 17.

Abbreviated Long Division

To abbreviate to a certain extent the work of long division the writing out of the products may be omitted, and the subtractions may be made mentally and the remainders only written down.

Divide 27815 by 31.

```
31 ) 27815 ( 897 8/31
      30
       22
        8
```

It is put down as for long division. Inspection shows that the first figure of the quotient is 8. Then we say 8 times 1 are 8 and subtracting this first figure of the product from the dividend we put down 0 with nothing to carry. Next 8 times 3 are 24, which subtracted from 27 leaves 3 which is put down. The 1 is taken down mentally from the dividend and it is seen that 31 goes into 301 9 times. Proceeding we have 9 times 1 are 9 and subtracting this 9 from the 1 of the dividend leaves 2 which is put down with 1 to carry. Then 9 times 3 are 27 and 1 to carry makes 28 which subtracted from 30 leaves 2, the last figure of the remainder. Taking down 5 mentally, 31 goes into 225 7 times. 7 times 31 are 217, which subtracted from 225 leaves a final remainder of 8, which is written in the quotient as numerator of a fraction $\frac{8}{31}$.

Italian Method of Long Division

What is termed the Italian method of doing long division consists in putting both divisor and quotient on the right hand of the dividend. It is just as good to put them on the left hand. The example shows the method. The advantage is that the divisor and quotient are in position to be multiplied to prove the correctness of the operation. The operation is done by the short method described above. 27 is the divisor and 42 is the quotient.

$$\begin{array}{r|l} 1134 & 27 \\ 5 & \text{—} \\ 00 & 42 \end{array}$$

Excess of Nines in Division

The proof of the correctness of a multiplication by excess of nines has been explained. The same can be applied to division. The excess of nines in the divisor multiplied by the excess of nines in the quotient, and the excess of nines in this product added to the excess of nines in the remainder will give the excess of nines in the dividend if the operation has been done correctly. Take the division of 9763 by 281; the quotient is 34 and the remainder is 209. The excess of nines in the divisor, 281 is 2; the excess of nines in the quotient, 34, is 7; the product of these, 2×7 is 14; the excess of nines in this product is 5; the excess of nines in the remainder, 209, is 2; the sum of 5 and 2 is 7, and we find the proof that the division is in all probability correct in the excess of nines in the dividend, which is also 7. While it is quite conceivable that the excess of nines would come out right when the division was wrong, it is exceedingly improbable; if they do come out wrong the division is certainly incorrect.

If the process be compared with the similar operation for the proof of multiplication, it will be seen that the one follows from the other.

The example is given here.

```
281 ) 9763 (34
      843
      ----
      1333
      1124
      ----
       209
```

Excess of 9's in the divisor, 281 is 2
Excess of 9's in the quotient, 34, is 7; $7 \times 2 = 14$; excess of 9's is.............................5
Excess of 9's in the remainder, 209 is 2, to be added to above 2

—

7

Excess of 9's in the dividend, 9763 is.............. 7

It follows that the division has been correctly performed because of the agreement in excess of nines.

Remainders in Division

When a number is used as a divisor of numbers indivisible by it without a remainder, the possible remainders are one less in number than itself.

Thus for 3, used as a divisor, when it divides a quantity giving a remainder, there can only be two possible remainders, 1 and 2. For the number four, used as a divisor, the only possible remainders are 1, 2 and 3. The same rule applies to all other divisors.

When the division is carried out beyond the decimal point, the remainder may give a continued or a complete decimal.

If divisions of numbers, indivisible by the divisors used, are carried out so as to give decimals in the quotient, interesting results are obtained.

With 2 as a divisor, the only possible remainder is .5.

With 3 as a divisor, there are two possible remainders, both continued decimals, .333 . . . and .666 . . . .

With 4 as a divisor, the remainders are .25, .5, and .75.

With 5 as divisor, the remainders are .2, .4, .6 and .8.

With 6 as a divisor, the remainder may be .5, or continued decimals ending in 3's or in 6's.

With 8 as a divisor, the remainders end always in 5, and are one, two or three figure decimals, .5, .25, .125, .375, .625 and .875.

This leaves the numbers 7 and 9 to be considered.

The remainders given by the number 7 are repetends, identical in all cases, except that the decimals begin with different numbers; the order of the component digits is the same. The decimal remainders for the six possible remainders of seven used as a divisor, are the following:

.142857....	.571428....
.285714....	.714285....
.428571....	.857142....

The first of these remainders is divisible by 7 with a remainder of 1, reducing to the same decimal; this is why it is a repetend. The quotient obtained by dividing it by 7 is .020408+, a rather curious succession of numbers.

The number 9 gives as remainders continued decimals, of which each one is simply the repetition indefinitely of

the remainder figure. Suppose we have to divide 253 by 9. The quotient will be 28 and 1 over, or a remainder of 1. To put this into decimals we simply repeat the remainder indefinitely, as a continued decimal, and the quotient is 28.111 . . . .

Suppose now that 290 is to be divided by 9; this gives a quotient of 32 and a remainder of 2; in decimals the remainder is 2 repeated indefinitely, as a continued decimal, and the quotient is 32.2222 . . . .

This covers single digit divisors. The reader can try other divisors *ad libitum,* and will obtain quite interesting results.

Divisibility of Numbers

A number divisible by 2 is called an even number; such numbers end in one of the even numbers—2, 4, 6, 8 or else in a cipher, 0.

A number is divisible by 3 when the sum of its digits is divisible by 3. Thus the number 123 has as digits $1+2+3=6$. As this sum, 6, is divisible by 3 it follows that 123 is divisible by 3. The digits of 252 add up to 9; $2+5+2=9$. As 9 is divisible by 3 it follows that 252 is also. The quotients of the two examples given are 41 and 84.

Any even number divisible by 3 is divisible by 6. Thus the number 252 is an even number because it ends in 2; it is divisible by 3; therefore it is divisible by 6; divided by 6 the quotient is 42.

A number is divisible by 4 when its last two digits are divisible by 4. The last two digits of 9824, namely 24, are divisible by 4, therefore the whole number is; on trying it the quotient is found to be 2456.

All numbers ending in 5 are divisible by it.

All numbers ending in 0 are divisible by 5.

All numbers ending in 25 are divisible by it.

A number the sum of whose even placed digits is equal to the sum of the odd ones is divisible by 11.

Thus the sum of the even digits of the number 1538779 is 20, and the sum of the odd digits is also 20; the number therefore is divisible by 11.

If the last three figures of a number are divisible by 8 then the whole number is divisible by it. This is because 1000 is divisible by 8, so whatever precedes the last three figures it eventually comes to adding one or more thousands of them, which of course does not affect their divisibility by 8. Take 128; this is divisible by 8; if we add 1000 we have 1128, which of course if divided by 8 gives 141. 128 divided by 8 gives 16 and 1000 divided by 8 adds 125 to it making $16 + 125 = 141$. No matter how many figures are put before any three figures divisible by eight it only amounts to putting so many 1000's before the three, and each 1000 is divisible by 8. This property is practical as well as useful, and is a parallel case to the similar property of 4.

792 is divisible by 8. Put any numbers before it—say 33. This gives 33792. Dividing we find that 8 goes into 33 4 times and 1 over, so now we have to divide our original number increased by 1000, for that is what the effect of carrying 1 is, it gives 1792 as the number to be divided. It divides by 8 without remainder.

When the difference between the sum of the odd and even placed digits of a number is divisible by 11, the whole number is also divisible by it.

Take the number 54912; its even digits are 1 and 4,

whose sum is 5; its odd digits are 5, 9, and 2, and their sum is 16; the difference between 5 and 16 is 11; therefore the whole number is divisible by 11; the quotient is 4992. Take the number 27192; here the even digits have the larger sum, but the same rule holds; the difference is divisible by 11 and the whole number is so also.

If a number is indivisible by 4 without a remainder, and we divide it by 4 and carry it out in decimals, the quotients will contain either the decimal .5, the decimal .25, or the decimal .75 and no other.

Thus $22 \div 4 = 5.5$; $25 \div 4 = 6.25$; $27 \div = 6.75$.

The above reduces in vulgar fractions to ¼, ²⁄₄, and ¾. An analogous rule obtains for all such divisions; the remainders for division by three will be ⅓ and ⅔; for division by five the remainders will be ⅕, ⅖, ⅗ and ⅘. The numerators of the remainders, when expressed as vulgar fractions will begin with one and go to one less than the divisor, and the divisor will be the denominator of the fractions.

The general statement of the above follows, together with its application to the single digits used as divisors of numbers giving remainders.

If the digits of any number are added together, their sum or the sum of the digits of their sum is the remainder which will be left on dividing the original number by 9.

Thus the sum of the digits of 875 is 20; the sum of the digits of 20 is 2; if we divide 875 by 9 the remainder will be 2, the sum of the digits of 20.

The sum of the digits of 962176 is 31; the sum of the digits of 31 is 4; if we divide the original number, 962176 by 9, the quotient will be 106908 with a remainder of 4, the sum of the digits of 31.

A number divisible by 2, 3, 4, 5, and 6 always with a remainder of 1, is divisible by 7 without any remainder.

The smallest of such numbers is 301. If it be divided by any of the five numbers cited, there will be a remainder of 1; if we divide it by 7, there will be no remainder.

Starting with 301 as a base, by successive additions of 420, other numbers of the same property can be obtained; such are 721, 1141, and others.

If the number 2519 is divided by any single number, it will give a remainder one less than the divisor. If divided by 2 the remainder will be 1; if divided by 3, the remainder will be 2 and so for the nine digits. It is the smallest number possessing this property.

Special Cases of Division

To divide a number by 5 multiply by 2 and cut off the last figure by a decimal point.

To divide 28 by 5, double it giving 56, and put the decimal point to the left of the last figure, 5.6 or $5\frac{6}{10}$. Or to take a larger number—714. To divide by 5 double it and place the decimal point to the left of the last figure, and we obtain 142.8.

To divide by 25 quadruple the dividend and cut off two figures.

Divide 1297 by 25. The product of 1297 multiplied by 4 is 5188, and placing the decimal point as directed we have as the quotient, 51.88.

This rule with a variation in the multiplier obtains for all powers of 5. For 125 as a divisor multiply by 8 and cut off three figures by the decimal point.

To divide any multiple of 11 by 11 proceed as follows: Put down the right-hand digit of the dividend as the

right-hand figure of the quotient. Subtract this from the next figure of the dividend, carrying 1 if necessary. The result of the subtraction gives the next figure of the quotient. The process is carried out to the end.

Divide 56408 by 11. Put down 8 as the right-hand figure of the quotient. Subtract 8 from the next figure, 0, of the dividend; this gives 2, the second figure of the quotient, with one to carry. Adding the 1 to be carried to the second figure of the quotient gives 3, and 3 subtracted from the next figure of the dividend, 4, gives 1, the third figure of the quotient, with nothing to carry. Finally this third figure of the quotient is subtracted from the next figure of the dividend, 6 giving 5, the fourth figure of the quotient, leaving 5 of the quotient to be subtracted from the last figure of the dividend, 5, leaving as the quotient, 5128, these being the figures found.

Dividing by Ninety-Nine

To divide by 99, add the two right hand figures of the dividend to the rest of the number and put it down under the original number. Do the same with the smaller number, and put it down under the other two. Keep up this process until 99 is left or a quantity less than 99. Cut off the two right hand figures of everyone of the quantities, either mentally or by drawing a vertical line; add up what is left on the left of the line; this is the quotient. The number at the bottom of the right hand of the line is the remainder; of course if the remainder is 99, 1 has to be added to the quotient.

Divide 869432 by 99.

The two right hand figures are 32; these are added to the remaining figures, thus: $8694 + 32 = 8726$. This

is the first quantity to be added to the original dividend. To get the next number we cut off from 8726 its two right hand numbers, 26 and add them to what is left of the number; we have thus: $87 + 26 = 113$. This is the second number to be added to our original dividend. The same operation is applied to 113; its two left hand digits are cut off and added to what is left; this gives $1 + 13 = 14$. As this is only a two digit quantity nothing more is to be done to it. We now set down the quantities obtained and draw the vertical line as below.

$$\begin{array}{r|l} 8694 & 32 \\ 87 & 26 \\ 1 & 13 \\ & 14 \\ \hline \end{array}$$

8782 with a remainder, 14.

This is the result of dividing 869432 by 99.

Divide 23661 by 99.

Proceeding exactly as before we have as our quantities 23661, $236 + 61 = 297$ and $2 + 97 = 99$; these we treat as before:

$$\begin{array}{r|l} 236 & 61 \\ 2 & 97 \\ & 99 \\ \hline \end{array}$$

238 with a remainder 99, giving 239 as the quotient.

Properties of the Number Three in Division

If we take any two numbers we will find that either their sum or their difference or one or both of the numbers is divisible by three. 19 and 17 seem rather hopeless, but their sum is 36, and 36 is divisible by 3.

Any number the sum of whose digits is divisible by 3 can be divided by 3 itself. A clue to the reason for this lies in the fact that if we multiply all the single digits by three we shall find the nine digits represented in the terminal figures. Three times runs—3, 6, 9, 12, 15, 18, 21, 24, 27—these are the products of 3 by the nine single numbers and the same nine numbers are there in the last places. Now take a number, say 74228115—the sum of its digits is 30, this is divisible by 3. Now try the number itself and its quotient when divided by 3 is 24742705 with no remainder. The numbers 7 and 9 also have the nine digits as the last figures in their multiplications by the nine first figures, but that does not confer this property on 7, but 9 possesses it.

Suppose a number is not divisible by 3, then the amount by which the sum of its digits exceeds the nearest lower multiple of 3 will be a number which if subtracted from the original one will leave it divisible by 3 without a remainder. Take the number 3983; the sum of its digits is 23; the next lower multiple of 3 is 21, and 2 therefore is the difference to be subtracted from the original number; carrying out the subtraction we get 3981, which is divisible by 3, the quotient being 1327 and there is no remainder.

Lewis Carrol's Short Cut

This is Lewis Carrol's short cut for dividing any multiple of 9 by 9. It is only a matter of interest, and in great measure such, because of its distinguished originator, the author of "Alice in Wonderland" and of "Alice in the Looking Glass"; the originator of the Wonderland arithmetical rules of Ambition, Distraction, Uglifi-

cation and Derision. (Alice in Wonderland, Chap. IX.)

Write down the number to be divided; put a cipher over the units figure and subtract the units figure from the cipher. The result is the units figure of the quotient. Then put this figure over the tens figure of the number to be divided and subtract the tens figure of the original number from it and put it down as the tens figure of the quotient. Write it also above the hundreds figure of the number and subtract the hundreds figure from it and put it down as the hundreds figure of the quotient and also put it over the thousands figure and so on until the number runs out.

Divide 36459 by 9.

Write out the dividend, 36459. Put a cipher over the right hand or units figure, subtract and put the figure down as the units figure of the quotient and also put it over the second or tens figure of the number, in this case, 5. The process is repeated according to the directions. The successive steps are given here:

10	510	0510	40510	
36459	36459	36459	36459	
51	051	4051	4051	the quotient.

Inspection of the last operation shows that it all is simply subtracting the dividend from ten times the quotient.

CHAPTER VI

FRACTIONS

Vulgar Fractions

Any quantity numerically less than unity is a fraction. A vulgar fraction is one expressed as a division; the divisor classifies the fraction, as being of the half class, the thirds class, the thirtieths class or any other class whatever. A definite one of these divisors, which are called denominators, is taken for each fraction—the figure expressing the dividend is called the numerator.

To write a fraction the numerator is placed above a short horizontal or diagonal line or bar, and the denominator is placed below the same bar.

$1/3$, $12/30$ are fractions; the first belongs to the class of thirds, as its denominator tells us, the next one belongs to the class of thirtieths as its denominator indicates. The numerator of the first fraction tells that only one of its class is taken, the numerator of the next one tells that twelve of its class are taken.

Meaning of the Fractional Bar

One most important thing to be realized about fractions is the true significance of the bar. It is a sign of division, just as truly as is the sign $\div$. $5/13$ may be expressed in words as five thirteenths or as five divided by thirteen; it could correctly be written $5 \div 13$.

The fractional bar may be and often is used as a sign of division. $125 \div 25 = 5$ can be written $\frac{125}{25} = 5$; both mean identically the same thing.

The fractional bar is sometimes an oblique one; this is merely a matter of taste or of convenience.

Although it is never used as the sign of division. a fraction might be written perfectly correctly as the indication or setting down of a calculation in division; $\frac{3}{16}$ might be written 16)3, for this is the carrying out to its full meaning of the fraction with its bar. If the division is completed the result may be expressed in decimal fraction.

Changing the Value of a Fraction

There are two ways of increasing the value of a fraction; one is to increase its numerator, the other to diminish its denominator. The reverse also is true, if the denominator is increased or the numerator diminished the value of the fraction will be diminished.

Thus $\frac{3}{7}$ is smaller than $\frac{4}{7}$ and also smaller than $\frac{3}{6}$; it is larger than $\frac{3}{8}$ or than $\frac{2}{7}$. The reader will observe how the numerators and denominators are increased and diminished in these examples.

Reducing Fractions to the Same Class or to a Common Denominator

If you want to add a gallon to a pint you must reduce them to the same class either to gallon or to pint class; the sum of the two is one and one eighth gallon or is nine points. To add two fractions they must be of the same class, that is to say must have the same denominator.

Addition and Subtraction of Fractions

To add $\frac{1}{2}$ to $\frac{1}{3}$ we must first multiply numerator and denominator of each fraction by the denominator of the

other; this gives $\frac{3}{6}$ and $\frac{2}{6}$, and the sum of these is $\frac{5}{6}$; the numerators are added because they tell how many there are of each—if there are two of them to be added to three of them the result is, of course, 5. The things added are sixths, and there are five of them.

For subtraction the reverse is carried out; the reduction to a common denominator is effected and one numerator is subtracted from the other one. $\frac{1}{3}$ subtracted from $\frac{1}{2}$ would give $\frac{1}{6}$.

The addition of fractions of different denominators can be expressed in words without reduction to a common denominator; the above addition can be called a half and a third. This is constantly done in adding fractions to whole numbers—two added to a half is usually expressed as two and a half. It might be expressed as $\frac{5}{2}$ or five halves.

Multiplication and Division of Fractions

If we multiply a number by a fraction whose numerator is 1, we can equally well divide the same number by the denominator. $2 \times \frac{1}{2}$ or $2 \div 2$ give the same result, namely, $\frac{2}{2}$ or 1.

Any number may be regarded as a fraction; as the denominator of a fraction expresses what may be called its class, and as a whole number refers to the class of units, if a whole number is written as a fraction its denominator must be 1. Thus any whole number can be written over the fractional bar as a numerator, with the denominator, 1, under the bar. 125 can be written $\frac{125}{1}$ for instance. As nothing is gained by doing this it is never done, just as the decimal point is omitted in writing whole numbers. As a matter of correctness the number

125 should have the decimal point expressed, 125., although it is always omitted, unless there is a decimal fraction to go there.

In the same way the bar and unit denominator are always omitted in writing whole numbers; nothing would be gained by putting them down.

As the fractional form indicates division, if we are to divide a number by another it can be done directly by division, or the divisor can be made the denominator of a fraction whose numerator is 1, and the dividend can be multiplied by the fraction.

Dividing by a number, or multiplying by a fraction having the number in question as a denominator and having 1 for its numerator, is identical.

If 25 is to be divided by 5 the operation may be effected by ordinary division giving 5, or it may be expressed as $25 \times 1/5$.

To multiply by a fraction multiply by its numerator and divide by its denominator.

Thus $25 \times 1/5$ is 25 multiplied by 1 and divided by 5, which gives $25/5$ which is 5.

To multiply a fraction by a whole number its numerator may be multiplied by the number. To multiply $1/5$ by 25 gives, if we allow this rule, $25/5$ exactly as above, as it should be.

Taking a fraction with a numerator greater than unity, say $2/4$ to multiply it by the same fraction as in the previous example, namely $1/2$, we can equally well divide its numerator by the denominator of the multiplier or by 2; this division can be done in another way; the denominator of the multiplicand, $2/4$, can be multiplied by 2; the first operation gives us $1/4$, the other operation gives $2/8$; both these are of identical value.

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The general rule for multiplication by a fraction having a numerator 1, is to multiply the denominator of the multiplicand by the denominator of the fractional multiplier.

Now suppose the fraction multiplier had a number greater than unity for its numerator; it is clear that it would be that many times larger, so to multiply by it, we should multiply the numerator of the multiplicand by the numerator of the fractional multiplier, or, what is the same thing, we should divide the denominator of the multiplicand by it. Next we should divide the numerator of the multiplicand by the denominator of the multiplier. The result is the same, but it is simpler to multiply the denominators and numerators together. Therefore to multiply two fractions, multiply the numerators together for a new numerator and the denominators for a new denominator.

A fraction can also be multiplied by a whole number by dividing its denominator by the multiplier. To multiply $1/5$ by 25 by this method, we would obtain the expression, $\frac{1}{5 \div 25}$ which is equal to $1 \div 1/5$ which is 5. This is the same result as that obtained by the other methods.

If a number is to be multiplied it means that the number is to be taken as many times as there are units in the multiplier. If the multiplier is a fraction, say $1/2$ it has one half a unit in it; if a number is multiplied by $1/2$ therefore it can be taken one half time; $10 \times 1/2 = 5$.

If a fraction is multiplied by another fraction the same principle applies. $1/2 \times 1/2$ means that $1/2$ is to be taken one half time, which gives of course $1/4$. If $1/2$ is to be multiplied by $3/4$ it must first be taken three times, because of the numerator, 3, and then one fourth time because of

the denominator, 4. If we take $\frac{1}{2}$ three times it gives $\frac{3}{2}$; if next we take $\frac{3}{2}$ one fourth time we have $\frac{3}{8}$, which is one fourth of $\frac{3}{2}$.

If the same operation of division or multiplication is effected on both numerator and denominator of a fraction, it is evident that the ratio of numerator to denominator will be unchanged.

$\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$ are all the same in value; the last two are obtained by multiplying numerator and denominator by 2 and by 3; the ratios are the same in all.

If we divide the denominator of a fraction by one tenth of itself and the numerator by the same the value of the fraction will be unchanged but we will obtain a fraction whose denominator will be a multiple of ten.

This is a clumsy way of reaching the desired result; it is easier to divide the numerator by the denominator with due regard to the decimal point. But as a matter of explaining it is well to try it the first way.

Conversion of Vulgar Fractions into Decimal Fractions

Suppose we divide the two elements of the fraction, $\frac{1}{2}$ by one tenth of the denominator. We have for the numerator $1 \div .2 = 5$, and for the denominator, $2 \div .2 = 10$, and the new fraction is $\frac{5}{10}$. The value is unchanged and as it is a decimal in its denominator, it can be written as a decimal fraction, .5.

The more direct way, and the way always used to get a decimal fraction from a vulgar fraction, is the direct division of the numerator by the denominator.

To reduce $\frac{1}{2}$ to a decimal we divide as follows: $2\underline{)\,1.0}$; the same result as obtained above in the more

.5

indirect way.

Every decimal has a denominator understood; it is 1 followed by as many ciphers as there are figures in the decimal, ciphers coming between the characteristic figure or figures and the decimal point counting as figures.

If the characteristic figures of a decimal are taken as whole numbers and are placed over the fractional bar with the denominator determined as above below the bar the decimal will be expressed as a vulgar fraction.

Thus .5 is the same in value and can be expressed as $5/10$; .05 as $5/100$, and so on.

This is one way of expressing a decimal as a vulgar fraction and it gives the way in which it is always expressed in words; .5 is called five tenths, .05 is called five hundredths.

Another way to reduce a decimal is different in its result; it gives a fraction of the same value, but one not having ten or a multiple of ten for a numerator. It is the reverse of the operation just described. It consists in expressing the decimal as a vulgar fraction, and then dividing the characteristic figure or figures of both numerator and denominator by the characteristic figures of the numerator.

.5 is expressible as $5/10$; dividing both numerator and denominator by 5 gives $1/2$, which is the value of the decimal.

We may also divide both parts of the fraction by any common divisor, that is to say by any number which will divide both parts without remainders. Take the decimal .75; this can be written $75/100$; both the numerator and denominator are divisible by 25; doing this gives the vulgar fraction, $3/4$, which is the value of .75. Or both might have been divided by 5 giving $15/20$, also of the same value as .75.

CHAPTER VII

THE DECIMAL POINT

Errors Due to Misplaced Decimal Point

No class of errors in arithmetic are so frequent as those due to the decimal point. Put in the wrong place the smallest error it can introduce is a multiplication or a division by ten, and the error may be larger to any extent whatever, but can never be less. There is a tendency to trust to one's understanding in the matter of the decimal point, and the everyday expression of percentage as an integral figure, instead of a decimal, is an example of a source of confusion.

Six per cent. is written 6%; it is fair to say that many never picture 6% as really indicating a multiple which is a decimal, .06 or a vulgar fraction, $\frac{6}{100}$ths.

Any percentage for purposes of calculation is completely and properly expressed as a decimal. Five per cent. of one hundred dollars is properly obtained by multiplying \$100 by .05, giving \$5.00 as the answer.

Addition of Decimals

In adding decimals the same rule is followed as in adding integral numbers. The first rule of addition is to put units under units and tens under tens and so on. On the right of the decimal point in putting down an addition of decimals and in carrying out the operation, put tenths under tenths and hundredths under hundredths and so to the last of the decimals in the numbers.

Follow the same rule you follow when adding up dollars and cents. A cent is one hundredth of a dollar and is written correctly \$.01; ten cents are the tenth of a dollar, \$.1, which however, as we never think of this amount as a dime, but always as ten cents, is written \$.10. The final 0 does not affect its value in any way whatever.

Now suppose five dollars and fifteen cents are to be added to four dollars and twenty five cents.

The addition is carried out with dollars under dollars and cents under cents.

$$\begin{array}{r} \$5.15 \\ 4.25 \\ \hline \$9.40 \end{array}$$

Suppose five and fifteen hundredths gallons of turpentine were to be mixed with four and twenty five hundredths gallons of linseed oil and that there was no shrinkage what would the total amount of the mixture be? The numbers would be put down precisely as above except that the dollar mark would be omitted. Units would go under units, which brings the 4 under the 5; then going to the right of the decimal point we put the 2 under the 1, because they are both tenths; 5 is put under 5 because they are both hundredths. If an error is made in putting the digits where they belong the result will be incorrect.

Although the cipher is always used in writing such amounts as ten cents or fifty cents, the latter can just as correctly be written \$.1 and \$.5 as \$.10 or \$.50. The sum of the last example could have been written \$9.4.

Ciphers never need be placed after the last digit of a decimal.

Suppose in some complicated calculation the above amount had had its decimal point misplaced, say one place wrong to the right—it would then have read $94.

This is an example of a frequent error in arithmetical work.

By following the rule of keeping units under units and tenths under tenths and so for all integers and for all decimals no misplacing of the decimal point should occur in addition.

No one would ever think of adding dollars to cents in the same column; precisely the same applies to decimals when added.

Subtraction of Decimals

The same applies to subtraction of decimals. To subtract one of the numbers of the last example from the other, they would be written down in exactly the same way; hundredths would have been subtracted from hundredths, tenths from tenths and units from units and the result or remainder would have been, .90 or .9, for both of these are the one and same; it is only a matter of convenience which way you select to write them.

Here a difference appears between decimals and integers in practice; as many ciphers as are convenient may be placed to the right of a decimal without changing its value and for one reason or another ciphers are often so placed, but it is not so often that ciphers are placed to the left of an integral number, although it is perfectly logical to do so.

Multiplication of Decimals

It seems a matter of carelessness to make an error in the decimal point in addition or subtraction; in multi-

plication and division it is not an impossibilty and it is not infrequent.

To multiply decimals or mixed numbers there is no necessity of putting units under units and tenths under tenths, although as a matter of discipline and good practice it is to be recommended.

The decimals in a product are equal to the sum of the decimals in the numbers multiplied. If in the multiplication a final o appears in the product, it counts as a decimal and should be written because it has to be taken into account in fixing the place of the decimal point.

Multiply 9.5 by 6.3, and also 7.5 by 5.2.

```
  9.5          7.5
  6.3          5.2
  ---          ---
  285          150
 570          375
 ----         ----
 59.85        39.00
```

Both multiplications are done regularly as in the case of integral numbers; each of the original numbers has one decimal, therefore their product must have two decimals. In the case of the first both are significant figures, and have a value in addition to fixing the place of the decimal point; in the case of the second product there are two ciphers; these are without numerical value and can just as well be omitted, but have to be put down to fix the place of the decimal point.

Placing the Decimal Point

A decimal may be defined as a multiple of ten or as some power of ten or as a dividend of some power of ten. If the latter it is called a decimal fraction.

The numbers, 190, 100 and the like are decimal numbers; the fractions .125, .076 and the like are decimal fractions.

It would be perfectly correct to write these decimal fractions, $^{125}/_{1000}$, $^{76}/_{1000}$; they would then be expressed as vulgar fractions and would be so termed, although strictly speaking they are decimals.

When an integral number is written it is customary to omit the decimal point. It is understood and is taken as being placed just to the right of the unit figure or cipher in the unit place of the number.

To multiply a whole number by 10, on account of the decimal point being left to be understood, all that is necessary is to place a cipher on the right. Thus $125 \times 10 = 1250$. If we had written the decimal point to the right of the original number, 125., which would have been perfectly correct, then to multiply by 10 we should have had to rub out the point, to put down the cipher directly after the 5, and if we wished we could place a new decimal point next to the 0, thus 1250., although it would be usually quite unnecessary.

But in the case of decimal fractions the decimal point can never be omitted.

To multiply a decimal fraction by 10 the decimal point is moved one place to the right; to multiply by 100 it is moved two places to the right. Thus $.125 \times 10 = 1.25$, which operation may be expressed in vulgar fractions thus: $\frac{1}{8} \times 10 = \frac{10}{8}$ or $1\frac{1}{4}$. In general any multiplication by a decimal multiplier is done by moving the decimal point as many places to the right as there are ciphers in the multiplier.

In the case of whole numbers the multiplication effected

by the annexing of ciphers on the right hand of the number, amounts to a shifting of the decimal point to the right; this would have to be done were it not customary to omit the decimal point in putting down whole numbers.

Division of Decimals

To divide a decimal fraction by a decimal number the reverse operation has to be done; the decimal point is moved one point to the left for every cipher in the divisor. Thus to divide .125 by 10 it is written .0125; to divide it by 100 write it .00125.

In the division of whole numbers by decimal divisors the same rule is followed. To divide 125, a whole number, by 10 the unexpressed decimal point is moved one place to the left, and we have $125 \div 10 = 12.5$. If it was to be divided by 100 it would become 1.25.

A number may lie upon both sides of the decimal point; it is then a mixed number. Such are 12.5 and 1.25, with whole numbers on the left of the decimal point and decimal fractions on the right. They may be written as improper fractions, $^{125}/_{10}$ and $^{125}/_{100}$, or as mixed numbers, $12^{5}/_{10}$ and $1^{25}/_{100}$., which reduce to $12\frac{1}{2}$ and $1\frac{1}{4}$.

In division of one number by another there is no need that the divisor shall be smaller than the dividend. The decimal point takes care of the relations of one to the other and enables the division to be carried out as far as desired if a division without remainder is impossible. A vulgar fraction is merely a statement of division to be done, the divisor being the denominator and usually larger than the numerator, which is the dividend.

The fraction, $\frac{1}{2}$, is merely the expression of a division to be done, the division of 1 by 2; the fraction $\frac{1}{25}$ is the expression of the division of 1 by 25.

If such divisions are carried out we obtain decimal fractions and here is where the decimal point must be watched. Dividing 1 by 2 gives .5 or five tenths; dividing 1 by 25 gives .04 or four one hundredths.

A mixed number is treated in exactly the same way. Take $12\frac{3}{5}$. Dividing 3 by 5 gives .6; this is annexed to the whole number and we have 12.6 or $12\frac{6}{10}$. Or we may write $12\frac{3}{5}$ as an improper fraction and carry out the division; $12\frac{3}{5} = \frac{63}{5} = 12.6$. All is done exactly as in the division of whole numbers but with close regard to the decimal point.

The insertion of a cipher between a decimal fraction or a whole number and the decimal point, or the removal of a cipher therefrom does affect its value as just described.

It is in the division of and by decimal fractions that the greatest liability to error occurs.

The rule is that the decimals in the quotient are equal to those in the dividend diminished by those in the divisor. If the divisor has more figures to the right of its decimal point than the dividend has, ciphers are to be annexed to the dividend to make the decimals equal in number.

Divide 1.25 by .25. Express it as a regular division and go through the operation. .25)1.25(5. As the decimals in both are equal there are none to go into the quotient.

Divide .125 by .25. We have .25).125(.5. Here there is one more decimal in the dividend than in the divisor, therefore there is one decimal in the quotient.

Divide 125 by .25. We annex two ciphers to the dividend so that it shall have as many decimals as the divisor; this gives: .25)125.00(500. As the decimals in the divisor are as many as those in the dividend there are no decimals to go into the quotient.

Although the annexing of the ciphers could have been dispensed with, and the decimal point could have been determined mentally, for the sake of certainty it is best to add ciphers until there are at least as many in the dividend as in the divisor. It is immaterial if there are more.

Divide .001 by 95. Add enough ciphers to make the dividend numerically equal to or greater than the divisor, irrespective of the decimals. Thus we have:

95).00100(.00001;

or carrying it still further:

95).0010000(.0000105.

In each case, as there are no decimals in the divisor, all that is necessary is to make the decimals in the quotient equal in number to those in the dividend.

CHAPTER VIII

INTEREST AND DISCOUNT. PERCENTAGE CALCULATIONS

Expression of Rate of Interest

The rate of interest is always expressed as an integral or mixed number, as five per cent, six per cent, six and a half per cent—5 p. c., 6 p. c., 6½ p. c.—as the case may be.

The correct way to write it is as a decimal, and perform the calculations into which it enters, with due regard to the decimal point. Thus six per cent is correctly written as .06—five per cent as .05 and so on.

Calculating Interest

To calculate 5 p. c. interest on $762.98 multiply the principal by .05.

$$\begin{array}{r} 762.98 \\ .05 \\ \hline 38.1490 \end{array} \qquad \begin{array}{r} 7.6298 \\ 5 \\ \hline 38.1490 \end{array} \qquad \begin{array}{r} 2\,)\,76.2980 \\ \hline 38.1490 \end{array}$$

As there are four decimals in the two numbers there must be four in their product. In the first example the operation is carried out as a regular multiplication. In the second the decimal point is shifted two places to the left and the multiplier is taken as the figure indicating the percentage rate. Both operations are the same in essentials; another way to do the calculation is to divide

by 2, as in the third example, after shifting the decimal point.

The principal as regards its integral figures was stated in dollars; the answer reads therefore \$38.15.

In interest calculations the decimal point should be kept in mind and used correctly—guessing at how many dollars and how many cents are in the result of a multiplication is frequently done; if it be remembered that a per cent is properly written as a decimal and if the placing of the decimal point is perfectly understood, the first method is the best way to work.

Since a per cent expresses really a stated number of hundredths it can be written as a vulgar fraction; six per cent can be written $6/100$. Interest calculations can be done with this expression.

Take \$97.63 as the principal and calculate interest at 7 p. c., using vulgar fractions

$$\begin{array}{r} 97.63 \\ 7 \\ \hline 100\;)\;683.41 \\ \hline \$6.83 \end{array}$$

Short Ways of Calculating Interest

If the 360-day year and the 30-day month is used in interest calculations, the process for 6% is simplicity itself. The interest for a year is 6%; the interest for a month is ½%; the interest for a day is $1/30$th of that for a month. With this as a starting point other standard rates of interest are calculated nearly as simply.

If the 365-day year is used then each interest may as well be calculated in the regular way.

Assuming that we do calculate the interest on any sum at the rate of 6%, we can easily derive the others from it.

For 3% divide the 6% amount by 2.

For 4% subtract $\frac{1}{3}$ from the 6% amount.

For $4\frac{1}{2}$% subtract $\frac{1}{4}$.

For 5% subtract $\frac{1}{6}$.

For $2\frac{1}{2}$% divide the last by 2.

For 7% add $\frac{1}{6}$.

For $7\frac{1}{2}$% add $\frac{1}{4}$.

For 8% add $\frac{1}{3}$.

For 9% add $\frac{1}{2}$.

There are other ways of doing interest calculations in a short way. It will be sufficient to give the methods, as the carrying of them out presents no difficulties.

For 4% divide the principal by 25. This is not a short way; in any case it is only an alternative method, and explains what 4% means.

For 5% divide the principal by 20.

For 2% divide the principal by 50.

Calculate the interest on \$379.68 at 4% and at $4\frac{1}{2}$% for three months and 10 days, taking the year at 360 days.

One month's interest at 6% is the half of 1%, and this is \$1.89; for three months it is \$5.67; ten days is taken as the third of a month so the interest at 6% for that period is \$0.63, and the sum of the two last is \$6.30. For 4% subtract one third or $6.30 \div 3$; this leaves \$4.20; for $4\frac{1}{2}$% subtract one quarter; this gives \$4.73. The answers are therefore \$4.20 and \$4.73.

Reduction of Interest Periods

One half of 1%, which is in decimals .005, is one month's interest at 6% on the 360 day basis for one month.

This is reduced to other rates by the methods just given. It can be reduced to other periods of time by addition, multiplication or division. One per cent is the interest for two months at 6%. The first column gives the reduction of one month to other periods; the second column the reduction for two months.

To reduce one month to				To reduce two months to			
120 days	multiply	by	4	120 days	multiply	by	2
90 "	"	"	3	90 "	"	"	1½
45 "	"	"	1½	45 "	"	"	¾
20 "	"	"	⅔	20 "	divide	"	3
15 "	divide	"	2	15 "	"	"	6

One Day's Interest

To calculate the interest for one day on any amount, the product of the amount by the interest is divided by the number of days in the year. If we assume the year to consist of 360 days the quotient obtained by dividing 360 by the interest rate divided by 100 will give a divisor by which if the principal be divided the quotent will be the interest on the amount in question for one day.

Suppose the interest at 6% is to be determined on $100.00 for one day. To obtain the general divisor divide the interest rate, 6, by 100. This gives .06. Dividing 360 by .06 gives 6000, the general divisor for 6%. Dividing $100.00 by 6000 gives .01666 or 1⅔ cents as the required interest.

The reason for using the above general divisor is that it saves one operation, the multiplication by the interest rate. To multiply by .06 and divide by 360 is the same thing as to divide by 6000—the operations are identical.

The above method of calculating interest is applied to

any number of days by multiplying the one day's interest thus determined by the number of days. Thus for ten days the interest on the above at the given rate would be 16⅔ cents—for thirty days it would be thirty times the amount for one day, or 1⅔ cents multiplied by 30, which gives 50 cents.

Divisors for Rates of Interest

General divisors for other rates of interest follow. They are calculated only for such rates as give integral divisors. If a fraction is in the divisor the process is of little or no practical value.

All the divisors are based on the 360 day year.

To use the table multiply the principal by the number of days and divide the product by the general divisor for the specified interest rate.

2½%	14400	6%	6000	12%	3000
3%	12000	8%	4500	15%	2400
4%	9000	9%	4000	16%	2250
4½%	8000	10%	3600	18%	2000
5%	7200			20%	1800

What is the interest on $1791.23 for 39 days at 4%?

Multiply the principal by the days—1791.23 × 39 = 69857.97, and divide the product by the general divisor for 4%; 69857.97 ÷ 9000 = 7.762 or $7.76.

Cancellation in Interest Calculations

Cancellation is often applicable in interest calculations, especially when the 365 day year is the basis. On the left of the line place the days in the year; on the right of the line place the interest rate, the principal and

the number of days for which the interest is to run.

What is the interest on \$1575.25 at 4% for 27 days; $\frac{365}{360}$ day year?

365	1575.25	360	1575.25
73	315.04		315.05
	.04	72	.04
	27		27
73) 340.2540 (4.67		72) 340.2540 (4.73	

The interest is \$4.67 on the 365 day year basis.

The interest is \$4.73 on the 360 day year basis.

The rate of interest paid for money, when bills due are not discounted, is not always realized by those failing to take advantage of discount for quick payment, or by those offering it. A usual system is to offer 1% or 2% discount if the bill is paid in ten days. Suppose the bill is paid in any case in thirty days; then for the extra twenty days the 1% represents a rate of about 18% per annum, and the 2% is a rate of 36% per annum. Sixty days net, less 2% ten days, is 14⅖% per annum. Even if we take a most moderate case we will find that thirty days net, less ½% for cash in ten days, is at the rate of 9% per annum.

Percentage Calculations

To calculate what per cent one number is of another divide the first number by the second, or divide the percentage number by the base.

What per cent is 99 of 108? Proceeding as above we find: $99 \div 108 = .9166$ or expressing it as a percentage, 91.66%.

What per cent of 99 is 108? Here we must divide the other way, for the percentage amount must always be divided by the base. Then $108 \div 99 = 1.0909$ or 109.09%.

Suppose a number is increased by a certain per cent. The question may be asked what per cent must be subtracted from the increased number to get the original number again. It will be a different per cent.

If 10% is added to 50 it will give 55. This is because 5 is 10% of 50. So to get 50 back again we must subtract 5 from our 55. But 5 is not 10% of 55. To find what per cent it is we must divide 5 by 55. $5 \div 55 =$ .0909 which is 9.09%.

A city has 100,000 inhabitants. Another city is 50% larger; how much smaller is the first than the last city? The larger city has 150,000 inhabitants; this is because 50,000 is 50% of 100,000, but as 50,000 is 33⅓% of 150,000 the smaller city is at one and the same time 33⅓% smaller than the 150,000 inhabitant city.

If the above operations are followed out with regard to the decimal point, it will be seen that the percentage sign, %, or the word per cent, applied to an integral or mixed number, has the effect of dividing it by 100, because any stated percentage is as many hundredths of the base number as there are significant figures in the percentage. Thus 6 per cent or 6% or .06 all indicate the same thing.

Approximate Calculations by Percentages

By using percentage corrections or fractional ones approximate results, near enough correctness to be used in practice, may often be obtained. Examples are very

numerous; the point is to get the principle well understood and then it can be applied to any new case.

A kilometer is about .62 of a mile. To reduce kilometers to miles we should multiply by .62. This is a two figure multiplication. It is easier to multiply by .6 and add $\frac{1}{30}$ or even 3%.

Reduce 23 kilometers to miles by the three methods cited above.

a. $23 \times .62 = 14.26$ miles.

b. $23 \times .6 = 13.8$
$13.8 + .46 = 14.26$ miles.

c. $23 \times .6 = 13.8$
$13.8 + .42 = 14.22$ miles.

In example *b* $\frac{1}{30}$ is added; in example *c* 3% is added; for all ordinary purposes one method is as good as the other.

A mile is equal to about 1.6 kilometers. Suppose we had to reduce 23 miles to kilometers.

We may multiply 23 by 1.6. The easy way to do this is to multiply by 8, giving 18.4 and then to multiply this by 2, giving 36.8 kilometers.

Or else add to the miles .6 of their number; $23 \times .6$ gives 13.8, which added to 23 gives 36.8 kilometers as before.

A meter is equal to a little more than 39 inches or a little more than $3\frac{1}{4}$ feet. To reduce meters to feet multiply by 3 and add $\frac{1}{12}$.

1100 meters are equal approximately to $(1100 \times 3) + (3300 \times \frac{1}{12})$ which is 3575 feet.

It may be done by percentage; we may multiply by 3 and add 8%. The result will be nearly the same; for the 1100 meters we should find $3300 + 8\% = 3564$. This is not so close, the percentage should be $8\frac{1}{3}\%$.

The fraction method is preferable and just as easy as the percentage method.

A kilogram is equal to 2.2 lbs. avds. To reduce kilograms to pounds multiply by 2 and add 10%. Thus 25 kilograms are equal to 50 lbs. plus 5 lbs. or 55 lbs.

The 10% may be added before the multiplication—$25 + 10\% = 27\frac{1}{2}$ and $27\frac{1}{2} \times 2 = 55$ just as before.

A pound avds. is equal to about .454 kilogram. Therefore to reduce pounds to kilograms divide by 2 and subtract $\frac{1}{10}$. For 25 pounds the first method gives: $25 \div 2 = 12.5$; and $12.5 - 1.2 = 11.3$ lbs. The ten per cent of 12.5 is taken as 1.2.

The English stone, used as a unit of weight, especially for men, is equal to 14 lbs. To reduce stones to pounds multiply by one half the number of pounds in a stone, namely by 7, and then multiply by 2. Thus if a man weighs 13 stones, to reduce the stones to pounds, multiply by 7, which gives 91, and multiply this by 2, which gives the exact weight in pounds, 182 lbs.

This last reduction is not approximate, as are all the preceding ones in the last page.

Then to reduce pounds to stones divide by 7 and then by 2 or the other way. To reduce 199 lbs. to stones; dividing by 2 gives $99\frac{1}{2}$ and this divided by 7 gives $14\frac{3}{14}$ stones, which is the answer.

CHAPTER IX

POWERS OF NUMBERS

Powers and Roots

The square of any number above 3 has two or more digits: the square of any number above 9 has three or more digits; there is no rule for determining when a series of values, with one more digit in each case than in those of the preceding series begins.

Powers and Roots of Decimal and of Mixed Numbers

The square of a decimal fraction must have at least two decimal places, and the number of decimal places must be an even number.

Thus the square of .2 is .04; the square of .4 is .16; the square of .11 is .0121.

It follows from the above that if we have to extract the square root of a single figure decimal, such as .4 or .9, we must add a cipher and use the number with cipher annexed as the first period for the extraction.

Thus the square root of .4 has to be taken as the square root of .40, or of .4000, or of .400000, and so on, carrying it out to as many places as desired, always annexing two ciphers, in accordance with the rule for extraction of the square root. The square root of .4 is .632+; that of .9 is .948+; the fact that on their face they appear to be single digit squares has nothing to do with the value of their true square roots.

Analogous laws for higher powers could be given, but the above is sufficient to illustrate the laws affecting the powers of decimal numbers.

It will be seen also that the powers of decimals, when such powers are greater than the first power, and the same applies to fractions, are smaller in value than the original quantities. For corresponding negative powers the reverse is the case; the powers for decimals and fractions are larger than the original quantities, for whole numbers negative powers are smaller.

Relations of Numbers and Square Roots of their Powers

The square root of any power of a number is the identical power of the square root of such number.

Take the number, 256, which is the 4th power of 4; its square root is 16, and 16 is the identical power of the square root of 4, because the square root of 4, which is 2, raised to the identical power, the 4th power, is equal to 16.

In the two double columns below, the numbers 4 and 9 are used as the bases for the illustration of this law.

The left hand column of each pair of columns contains the consecutive powers of 4 and of 9, the series being carried in each case up to the 7th power. The right hand columns of each pair contain the square roots of the powers in question. It will be seen that these roots are the identical powers of the square roots of 4 and of 9 respectively.

4	2	9	3
16	4	81	9
64	8	729	27
256	16	6561	81
1024	32	59049	243
4096	64	581441	729
16384	128	4782969	2187

In the left hand column take the cube of 4, which is 64. Its square root is 8, which is placed in the adjoining column in a line with 64. Now just as 64 is the cube of 4, so is its square root, 8, the cube of the square root of the original number, 4. The square root of 4 is 2 and 8 is the cube of 2.

Turning to the left hand column of the powers of 8, we find the 7th power of 9 to be 4782969; its square root is 2187, and 2187 is also the 7th power of the square root of the original number, 9; it is the square root of 3. The same will be found to apply to all the powers in the two columns, and the reader may try other numbers to any extent.

Terminal Digits of Squares of Numbers

No exact square ends in 2, 3, 7 or 8. A number ending in one of the other five digits may be a perfect square; if it ends in one of these four it cannot be. An even square may end in any one of the other digits, 1, 4, 5, 6 or 9 followed by an even number of ciphers. Thus 302500 is a perfect square; its root is 550. This gives a way of telling that a number has no perfect square root; it is a pity that it does not go a little further and tell that a number is a true square, but it does not.

If a square of a number ends in 4 the figure immediately preceding 4, the last figure but one, must be an even number or a cipher. Thus the square of 98 is 9604, the square of 62 is 3844.

If a square ends in an odd figure the figure preceding it must be an even one. The squares of 5, 7 and 9 are examples, 25, 49 and 81 respectively.

If a square ends in any even number except 4, the last figure but one will be an odd one. Thus the square of 16 is 256.

If a square ends in 5 the figure 2 will precede the five; in other words such a square will end in 25. Thus the square of 15 is 225.

A square may end in two ciphers or in two 4's; otherwise it cannot end in a doubled number. The square of 12 is an example of the doubling—it is 144. The square of any number ending in 0 ends in two ciphers.

No square may end in a single cipher or in an odd number of ciphers; it may end in any even number of them.

A square may end in three 4's; it cannot end in any other three figures. Thus the square of 462 is 213444.

The square of a number ending in one or more ciphers ends in twice the number of ciphers in question. The square of 90 is 8100; the square of 700 is 490000.

A Curious Fraction

The fractional quantity, $\frac{41}{12}$, if squared, gives the fraction, $\frac{1681}{144}$. The latter fraction possesses the curious property that if it is increased or diminished by 5, the sum in the one case and the difference in the other case will both be perfect squares.

Reducing 5 to the fractional form and with a denominator, 144, gives $\frac{720}{144}$, which is in shape to be added or subtracted from the other fraction. If we add it we have $\frac{2401}{144}$, whose square root is $\frac{49}{12}$, and if we subtract it the resulting fraction is $\frac{961}{144}$, whose square root is $\frac{31}{12}$.

Circular Numbers

The numbers 5 and 6 are called circular numbers. The property which is appealed to as a basis for the name or title is the fact that their squares, cubes and all other powers end respectively in the numbers, 5 and 6. All powers of 5 end in 5 and all powers of 6 end in 6.

Properties of Squares

Every number squared, as it is, or else when 1 is subtracted from it, is divisible by 3 and by 4. Try such squares as 81, 49, for cases where 1 has to be subtracted; 64, 144, and many others are divisible without the subtraction. Again every squared number is divisible by 5 if it is increased by 1, or if this does not make it divisible then it will be when diminished by 1. Try it on the numbers above and on others.

Some square numbers are odd ones, such as 81, 49 and the like. If from any odd square we subtract 1 the remainder will be divisible by 8. Thus 729 is an odd square number, it is the square of 27; subtract 1 and we have 728, which is 91×8.

If the sum of the squares of two numbers is a perfect square, then the product of the two numbers is divisible by 6. This would be a very useful property of squares if it only worked the other or both ways—but it does not. Thus the product of 6 and 9 is divisible by 6, but the sum of their squares is 117, which is not a perfect square. Here is where it does not apply.

If the difference of the squares of two numbers is a perfect square the sum and difference of the original numbers are squares or the double of squares. Try it

with 6 and 10, and then with 12 and 13. Thus $10^2 - 6^2 = 64$, which is a perfect square; the sum of 10 and 6 is 16 and the difference is 4, both of which are perfect squares. Such numbers are not particularly numerous.

Write in a line the squares of the natural numbers, 1, 2, 3 and so on. Under them write the series 1, 3, 5, 7... Then the squares will be obtained by adding the series numbers to the squares one by one.

a.	Natural numbers		1,	2,	3,	4,	5,....
b.	Their squares		1,	4,	9,	16,	25,....
c.	The series	1,	3,	5,	7,	9,	11,....
d.	Sum of *b* and *c*	1,	4,	9,	16,	25,....	

It will be seen that line *d* and line *b* are identical.

Property of the Square of Two

The number 2 is the only integral number whose square is equal to its double; this amounts to saying that $2 \times 2 = 2 \times 2$, taking one multiplication as the expression of squaring, and the other as the expression of simple multiplication.

Fermat's Last Theorem

What is known as Fermat's last theorem may be expressed as follows: There are no three integral numbers so related that, if all are raised to the same power, and the power is higher than the square, the sum of any two will be equal to the third. We know and have seen that there are any number of quantities the sum of the two squares of two of which are equal to the square of the third, but if we go above the squares of numbers no similar relation can be found to exist.

Properties of Cubes

The following property of the cubes of four successive numbers is analogous to the relation of the squares, $3^2 + 4^2 = 5^2$. The property refers to the numbers 3, 4, 5, and 6, and is expressed as below:

$$3^3 + 4^3 + 5^3 = 6^3.$$

The volume of a cube is equal to the cube of one of its sides. To solve the classic problem of the duplication of the cube by simple arithmetic, which of course was not contemplated in the original problem, we must calculate the length of the edge of the second cube, such that its cube will be twice the cube of the edge of the smaller cube. The relation of the edges must be as $\sqrt[3]{2}$ is to 1. No integral numbers can be found giving this relation.

Orders of Differences of Powers

Suppose a row of numbers is set down on a line, and that on the line below them the successive differences are placed, each difference naturally below the space that comes between the two numbers appertaining to it. This is called the first order of differences, and there will be one less difference than the original numbers. Next if the rows of differences are subtracted from each other the new row of figures is the second order of differences, and it will again be one less in number than its parent row. Now try this with the squares of numbers. The square of 1 is 1, that of 2 is 4 and so on.

Squares	1	4	9	16	25	36	49	64	&c.
First order	3	5	7	9	11	13	15		
Second order	2	2	2	2	2	2	2		

The second order of differences of successive squares is always 2, no matter how large the squares may be. The difference, 2, is the product of 1, the first and only order of differences between the natural numbers, multiplied by 2, the latter the exponent of the second power or square of any number. For cubes we find that a uniform series of differences is reached in the third order of differences.

Cubes	1		8		27		64		125		216		&c.
First order		7		19		37		61		91			
Second order			12		18		24		30				
Third order				6		6		6					

The difference, 6, is the product of 1 by 2 by 3—the continued product of the three exponents of the first, second and third powers. To get the final difference of the fourth powers of numbers the same process may be carried out, or what is simpler take the continued product of the exponents just as before. This gives for the fourth powers 1 by 2 by 3 by 4, which is 24; for the fifth powers the fifth order of differences will be 1 by 2 by 3 by 4 by 5, which is 120.

Powers in Progressions

The sums of any number of the odd numbers beginning with 1 and going on in regular succession gives a series of square numbers. A number of the additions are given here; they can be carried to any desired extent. The addition must always start with 1 and no odd number can be omitted from the series.

$$1+3=4$$
$$1+3+5=9$$
$$1+3+5+7=16$$
$$1+3+5+7+9=25$$
$$1+3+5+7+9+11=36$$
$$1+3+5+7+9+11+13=49$$
$$1+3+5+7+9+11+13+15=64$$
$$1+3+5+7+9+11+13+15+17=81$$

All the sums are perfect squares. The same process can be carried out as far as desired. The number of the quantities added tells the square root. Thus the addition of seven quantities gives a number whose square root is 7, and so for all other additions by this method.

Taking the number 3 as a starting point if we add together the first two uneven or odd numbers, namely 3 and 5 we have the cube of the number of digits used. We used the digits 3 and 5; adding them together gives 8; they are two in number and 8 is the cube of 2. Now take the next three and they will give the cube of 3, which is 27, thus: $7+9+11=27$. Now take the next four, and we shall get the cube of four, which is 64, thus: $13+15+17+19=64$.

This relationship of the odd numbers to the successive cubes holds good as far as we wish to go.

Here is another curious progression. If the cubes of the natural numbers in their regular order are written out, the addition of the successive cubes will give a series of squares of the series 3, 6, 10, 15, . . .

Natural numbers,	1,	2,	3,	4,	5,....
Their cubes	1,	8,	27,	64,	125,....
Additions, all squares,		9,	36,	100,	225....
Roots of the additions,		3,	6,	10,	15,....

Thus taking the second line and adding consecutively we find that 1 and 8 give 9 of the line below; then 1 and 8 and 27 of the second line give 36 of the third line and so for the rest or as far as it may be carried, and the roots of the squares of the third line give the fourth line, which is the series referred to in the text.

Relations of Squares of Two Numbers

Take any odd number and divide it into two parts such that they will differ by unity. These two numbers will be the roots of two squares which will differ in amount by the original odd number. Thus take the number 13. It can be divided into two parts, 6 and 7. Squaring these we get 36 and 49, whose difference is the original number, namely 13. The same applies to any odd number whatever; 5 divides into 2 and 3, and the difference of their squares 4 and 9 is 5, the original number.

Many other curious relations can be traced out. Any series of even numbers beginning with 12 and going on with increments of 4, 12, 16, 20, 24 and so on—is to be divided by 2. The number thus obtained is to be divided into two parts differing by 2. These numbers squared will give two numbers whose difference will give the original number. Take 16 for example; divide by 2 giving 8. From 8 we obtain two numbers differing by 2, namely 5 and 3, whose sum is 8. Squaring these we get 9 and 25, whose difference is 16, the original number.

This cannot be done with even numbers indivisible by 4, because such numbers when divided by 2 give odd numbers, and these cannot give the two aliquot parts differing from each other by 2. Try it for yourself.

A well known problem is the following: Find a num-

ber such that if 12 and 25 be successively added to it, the results will be square numbers. It is evident that the difference between the two squares will be 25 — 12 or 13. This we divide into two numbers differing by one and squaring these we have the numbers sought for. The two numbers into which we divide 13 are 6 and 7, their squares are 36 and 49, and the number asked for in the problem is 36 — 12 or 49 — 25, either subtraction giving 24, which is the answer.

The process can be carried to other differences. Take the series of numbers differing by multiples of 5. What squares differ by 35 may be asked? Divide by 5 which gives 7; divide in two parts differing by 5, namely 1 and 6; squaring these gives 1 and 36, two square numbers differing by 35.

The number 35 could have been made to give another answer by treating it as an odd number simply. Thus divide it into two parts differing by 1; 17 and 18; square these which gives 289 and 324, both squares and differing by 35 also.

A general rule is when you have the choice of two numbers to divide with, divide by the smaller. Thus we got an answer by dividing 35 by the smaller number, 5; had we divided by 7, which is the other divisor, and the larger one, we should have failed to get an answer. Also divide even numbers by even ones and odd ones by odd.

The Progression of Cubes

Take the cubes of numbers in their numerical order—the progression of cubes it is called—and add them consecutively, putting down each number as obtained. The results will be perfect squares as below.

Cubes—	1	8	27	64	125	216	&c.
Additions—	1	9	36	100	225	441	
Square roots—	1	3	6	10	15	21	

The set of square roots constitute a true progression with a difference increasing by unity, and the same square roots are the sum of the cube roots of the upper row of numbers. These cube roots are:

1 2 3 4 5 6

and their sums give the row of square roots as above. Thus 1 and 2 are 3, the second of the row of square roots, and 1 and 2 and 3 are 6, the third figure and so on.

A Curious Property of Two Squares

Here is another odd property of squares. Take any two numbers, one above and one below 25 and equally removed therefrom—say 14 and 36. The difference between each of them and 25 is 11. Then the difference of their squares will be 1100 and their squares will end in the same two numbers. We give the calculation for this and two other pairs.

$(36) = 1296$	$(44) = 1936$	$(29) = 841$
$(14) = 196$	$(6) = 36$	$(21) = 441$
1100	1900	400

The first two figures of these remainders are differences between the numbers and 25; 14 and 11 are 25; and 36 minus 11 are 25; and so for the others. The two numbers of the second pair differ by 19 each from 25,

and the numbers of the third pair differ by 4. The reoder can try the same calculations with 50 and with 75 as base numbers and see the relations.

The Sum of Two Squares

If the product of 4 by a number is such that the addition of unity thereto produces a prime number, then such prime number and any power of it is the sum of two squares.

Thus take the product of 4×3 which is 12 and add 1 thereto. This gives 13, a prime number, formed by adding unity to a product of 4. Thirteen is the sum of $9 + 4$ both square numbers. Now take a power of 13; $13^2 = 169 = 12^2 + 5^2$, again the sum of two squares. Next take 4×10 and add 1 thereto; this gives 41, a prime number and again the sum of two squares, 5^2 and 4^2. Next square this: it gives 1681, the sum of two squares, 40^2 and 9^2; we may cube it next; $41^3 = 68921$, and this is the sum of two squares, 236^2 and 115^2.

If two numbers are such that the sum of their squares is also a square, then the product of the two numbers is divisible by 6.

The numbers 3 and 4 have as squares 9 and 16; the sum of these squares is 25, which is also a square; therefore the product of the two original numbers is divisible by 6; this product is $3 \times 4 = 12$, into which 6 goes twice without remainder.

To find two such numbers, the sum of whose squares shall be a square take any two numbers and multiply them together. Twice their product will be one of the numbers, and the difference of their squares will be the other

Take 4 and 7 as the numbers to start with. Their

product is 28, and twice this is 56; this is one of the numbers to be found. The difference of their squares is $49 - 16 = 33$, which is the other number sought. The square of 56 is 3136; the square of 33 is 1089; the sum of the two figures is 4225, which is a perfect square, whose root is 65.

The Square of the Hypotheneuse

There is one right angle triangle which is very well known, and which has long been used for laying off square work, such as foundations of buildings, fence corners and the like. This triangle is one whose sides are in the proportion of 3 to 4 to 5. Thus for the corner of a field we may take 10 feet as the unit. Then if we measure off 30 feet on a side known to be correct, and then with a 40 foot string scratch an arc of a circle from the corner end of the field as a centre, and next with the outer end of the 40 foot line as a center and a 50 foot string strike an arc of a circle intersecting the other arc, a line from the intersection of the two arcs to the corner of the lot will be a right angle. A square can be extemporized from three laths on the same principle. One lath is cut a little over three feet in length, the others a little over four and five feet respectively. Holes are bored near the ends of the laths, exactly on the axis, and exactly three, four and five feet apart. Joining them by tightly fitting wire nails you will have an accurate square.

The following is an elegant rule for laying out any kind of right angle triangle. Take any two numbers. For one of the sides of the triangle take twice the product of the two; for the other side, the difference of the

squares of the numbers; and for the hypotheneuse the sum of their squares.

Suppose we take 2 and 7 as the numbers. Twice the product of the two is $2 \times 7 \times 2 = 28$. This is one side. The difference of their squares is $49 - 4 = 45$. This is the other side. Then for the hypotheneuse we take the sum of the squares—$4 + 49 = 53$.

By trying different numbers all sorts of right angle triangles can be calculated. It is obvious that the numbers taken must be different ones. Thus try 3 and 7 as the numbers and the resulting triangle will be almost equilateral. 4 and 7 give a triangle whose short side is almost exactly one-half the hypotheneuse.

The formation of the following series of mixed numbers hardly needs explanation. The series is $1\frac{1}{3}$, $2\frac{2}{5}$, $3\frac{3}{7}$, $4\frac{4}{9}$ . . . and so on as far as desired. The integral numbers and the numerators of the fractions are successively increased by the addition of 1; the denominators of the fractions are successively increased by the addition of 2. Now if these numbers are converted into improper fractions they will run as follows: $\frac{4}{3}$, $\frac{12}{5}$, $\frac{24}{7}$, $\frac{40}{9}$. . . . These improper fractions give the sides of perfect right angle triangles, that is to say of right angle triangles with integral number sides.

Taking the fraction $\frac{24}{7}$; this gives one side of the triangle as 7, the other side as 40 and the hypotheneuse as the square root of the sum of the squares of 7 and of 40. The sum of these squares is 1681, and the square root of 1681 is 41, which is the hypotheneuse of the triangle in question.

If we try the same with any fraction of the series the sum of the squares of its numerator and denominator will give a perfect square.

Approximations to Isosceles Right Angle Triangles

There are a few cases where the sum of the squares of successive numbers gives a perfect square. Such numbers are 3 and 4, 20 and 21, 119 and 120. There are six more pairs up to the pair 803760 and 803761, it is said. These numbers give the closest approach to isosceles right angle triangles, as no right angle triangle with equal sides exists.

Short Way of Raising to High Powers

If the exponent of a power to which a number is to be raised can be factored, the value of the power can be found by raising the number first to the power indicated by one of the factors, then raising this power to that of another factor until the factors are expended.

Suppose 2 is to be raised to the 9th power. 9 can be factored, it is equal to 3×3. Therefore we can raise 2 to the cube, which gives 8, and raise 8 to the cube (which is the other factor of 9), giving 512; this is the value of 2^9 or what is the same thing $2^{3 \times 3}$.

Suppose 7 is to be raised to the 12th power. It is much easier to factor the exponent, 12, taking it as $2 \times 2 \times 3$; to raise 7 to the second power, 49; to raise 49 to the second power, 2401; and to cube this, giving the value required, 13,841,287,201, than to multiply 7 the eleven times required to do it directly.

Extraction of Roots of High Powers

The more important application of this priciple is in the extraction of roots. If a root whose index or exponent is factorable is to be extracted, all that is necessary

to do, is to successively extract the roots indicated by the factoring.

If the sixth root of a number is to be extracted at one operation it is a long process, involving tentation. But if the square root is first extracted and the cube root of the square root extracted this gives the sixth root without the tedious extraction of a high root.

For the extraction of roots, whose indices are prime numbers this method is inapplicable.

Short Method for Calculating Squares

The following method of calculating the squares of numbers is available up to 100. Beyond that it could still be used by a good computator, but it is especially recommended for numbers under the century mark. It is based on the principle proved in algebra, that the product of the sum and difference of two numbers is equal to the difference of their squares. Thus $29 + 1$ multiplied by $29 - 1$ gives $(29)^2 - 1$. If therefore we multiply 30 by 28, for these are the values of $29 + 1$ and $29 - 1$, and if to their product 1 is added the result will be the square of 29. It is easier to multiply 28 by 30 than to directly square 29.

To square a number proceed as follows:

Add to the number or subtract from it a quantity which will produce a decimal number, such as 20 or the nearest product of units by 10. This gives a single digit to multiply by, which is very easy. Then subtracting the same quantity from the original number, multiply the two new numbers together and to their product add the square of the number added or subtracted.

Square 79.

Add and subtract 1; this gives 78 and 80; $78 \times 80 = 6240$, and adding the square of the increment, which in this case is 1, the square of 79 is the result, 6241.

Square 67.

Here we add and subtract 3, obtaining as our working numbers, 64 and 70. Multiplying 64 by 70 gives 4480, and to this must be added the square of the increment, 3; this square is 9 and $4480 + 9 = 4489$, which is the square of 67.

The clue to the process is by addition or subtraction to get a decimal number for one of the multipliers.

The next example shows the use of subtraction to get the desired decimal number.

Square 54.

Subtracting and adding 4 gives the two multipliers as we may call them, 50 and 58. It will be observed that the decimal number this time is obtained by subtraction. We have therefore $50 \times 58 = 2900$, and adding the square of the increment, 16, we have the square of 54, which is 2916.

Subtraction is used to get the two working numbers when the smaller increment is required to give the decimal number. It is only a matter of convenience whether addition or subtraction is employed.

Short Way of Calculating Squares of Large Numbers

The square of the sum of two numbers is proved in algebra to be equal to the sum of the square of the first added to the square of the second and to twice the product of the first by the second. The law is best illustrated by a couple of examples.

$$(93)^2=(90+3)^2=(90)^2+3^2+2(90\times 3)$$
$$=8100+9+540=8649$$
$$(24)^2=(20+4)^2=(20)^2+4^2+2(20\times 4)$$
$$=400+16+160=576$$

By a little practice this method becomes very easy of execution, and it can often be used to advantage as a method of calculating squares or as a way of proving results.

When applied to mixed numbers, such as $5\frac{1}{2}$, $6\frac{1}{4}$ and the like, a general rule may be thus stated: Add to the integral number double the fraction, and multiply this by the integral number; add to the sum the square of the fraction.

To calculate the square of $9\frac{1}{3}$ by the above rule: Add to the integral number, which is 9, twice the fraction, giving $9\frac{2}{3}$; multiply this by 9; this gives $9\times 9\frac{2}{3}=87$; adding the square of the fraction, $\frac{1}{9}$, we have $(9\frac{1}{3})^2=87\frac{1}{9}$.

This squaring is particularly easy because 9 is divisible by 3. If $8\frac{1}{3}$ is raised to the square by this process it will not be quite as simple, although there is no difficulty about it; thus: $8\times 8\frac{2}{3}=69\frac{1}{3}$, and adding the square of the fraction, which is $\frac{1}{9}$, gives $69\frac{1}{3}+\frac{1}{9}=69\frac{4}{9}$, which is the square in question.

Aliquot parts apply often here. 625 may be taken as $6\frac{1}{4}$. The method is applied with due regard to the decimal point; first we get $6\times 6\frac{1}{2}=39$; to this is added the square of $\frac{1}{4}$ which is $\frac{1}{16}=.0625$, and the final result is 390625, for the decimal point disappears as the original number contained none.

Various Ways of Squaring Numbers

If a number is divisible by 2, by 3 or by 5, its squaring may be simplified in many cases by dividing it by one of these numbers, squaring the quotient and multiplying by the square of 2 or of 3 according to which was the divisor we employed.

To square 33; dividing by 3 and squaring the quotient gives 121; this must be multiplied by 3^2, which is 9; $121 \times 9 = 1{,}089$.

To square 36, proceed exactly as in the last case; $36 \div 3 = 12$; squaring 12 gives 144, and this multiplied by 9 gives 1,296.

Now take a number ending in 5—say 45. Dividing by 5 gives 9, and squaring 9 gives 81, and $81 \times 25 = 2{,}025$. Here of course we multiply by 100 and divide by 4; it can be done mentally and put down directly on the paper.

If the square of a number is known the square of the number which is one greater than itself, is found by adding to this square its own square root and the number which is one greater than its own square root.

Thus to determine the square of 13; the square of 12 is 144; to it add its own square root, 12, and the number which is one greater, 13; this gives $144 + 12 + 13 = 169$, which is the square of the number one greater than 12, or the square of 13.

The product of two numbers increased by the square of half their difference is equal to the square of the intermediate number.

Thus the product of 24 and 25 is 600; their difference is 1, and the square of ½, or the square of half their difference, is ¼, the square of the intermediate number is 600¼; this number is 24½.

Again the product of 20 by 80 is 1600; half the difference is 30; if its square is added to 1600, it gives $1600 + 900 = 2500$, which is the square of the intermediate number, or the square of 50.

To square a number ending in ½ or what is the same thing for this method, one ending in 5, proceed as follows:

In the first case multiply the integral portion of the number by the next higher integral; add to the product ¼, and the result is the square of the original mixed number.

Thus the square of 8½ is $(8 \times 9) + \frac{1}{4} = 72\frac{1}{4}$.

If the number ends in 5, the 5 is taken as representing the ½ of the first case; thus the square of 65 is $(60 \times 70) + 25 = 4225$.

This rule is often very convenient.

To square any number between 25 and 75, subtract 25 from the number, multiply the remainder by 100 and add the square of the difference between the number and 50.

Thus to square 46 proceed as follows:

$$\begin{aligned} 46 - 25 &= 21, \text{ and } 21 \times 100 = 2100 \\ 50 - 46 &= 4, \text{ and } 4^2 = 16 \\ \hline & \qquad\qquad 46^2 = 2116 \end{aligned}$$

To square any number made up entirely of 9's, proceed thus:

Put down 1 for the right-hand figure; on its left put ciphers one less than the 9's in the original number; next put an 8, and finally 9's, one less in number than the 9's in the original number. By this rule we can write out the square of any such number at once; thus the square of 9999 is 99980001; the square of 99999 is 9999800001.

McGiffert's Methods of Squaring Numbers

The following method of squaring numbers between 40 and 60 is due to Prof. James McGiffert of the Rennselaer Polytechnic Institue.

The difference between the number and 50 is called the complement. If the number exceeds 50 then the complement is added to 25 and the square of the complement is annexed, not added, to the result. Suppose 57 is to be squared. The difference between 57 and 50 is 7, this is the complement. Carrying out the rule we have: $25 + 7 = 32$; annexing the square of the complement or 49 to 32 we have as the square of 57 the number 3249.

If the number is less than 50 the complement is subtracted and the square is annexed as above. Thus the complement of 44 is 6. $25 - 6 = 19$; the square of the complement is 36; annexing this to 19 we have 1936, the square of 44.

Except for the one variation (addition or subtraction) the processes are identical and are based on the rule for the square of the sum and difference of numbers—for the squaring of binomials, to use an algebraic term. The rule will operate on any numbers but it is most advantageous within the range given. The reason it comes so nicely is because one of the numbers is always 50, whose square is 2500 and the second number, namely the complement, is a monomial which is a number less than 10, whose square is in the multiplication table, i. e. whose square is known to everyone. Let us hope so.

If a number is doubled the square of the new number will be four times that of the original. By multiplying any number between 21 and 29 by 2, we can apply the

above rule, and then dividing by 4 will give the square of the original number. Thus take 27. Twice 27 is 54. Then $25 + 4$ with 16 annexed is 2916, and this divided by 4 gives 729, the square of 27.

If we divide any even number from 82 to 98 by 2 and apply the rule, it will give us one quarter of the square of the original number.

This process can be extended to apply to all numbers under 100, if the calculator will only learn the squares of the numbers from 13 to 24, if the doubling process be used to supplement the method for numbers under 25, and the halving process for numbers above 60. Professor McGiffert says these squares are easily learned. The reader is advised to try it.

For the square of a number between 100 and 110, add the complement to the number and annex the square of the complement. Of course if you have learned the squares as just recommended this rule will apply to any number above 100.

Take 109; the complement is 9. Add this to the number—$109 + 9 = 118$; to this annex the square of the complement—$9^2 = 81$—and we have 11881, as the square of 109, which is correct. If the square of the complement has only one digit a cipher must go before it, between it and the sum of the number and its complement, thus: $103^2 = 103 + 3$ with 3^2 preceded by 0 annexed, giving 10609 as the square of 103.

Between 90 and 100 the same rule applies except that you are to subtract the complement from the number. Thus to square 95 subtract the complement which is 5 and annex the square of the complement, 25, giving 9025, the square of 95.

For numbers above 250, add the complement, annex a cipher and divide by four, and annex the square of the complement. $259 = \frac{(259 + 9) \times 10}{4}$ annex 81 giving 67081. Multiplying by 10 is the same as annexing a cipher.

For numbers below 250 proceed as above but subtract the complement.

If the square of the complement contains three figures be sure to set it only two places to the right. Take 61 whose complement is 11. Following the first rule we add the complement to 25 giving 36; the square of the complement is 121, which is annexed with the 1 under the 6 of 36, thus: $\begin{array}{l} 36 \\ \underline{\ \ 121,} \\ 3721 \end{array}$ 3721 is the square of 61.

Prof. McGiffert's methods it will be seen apply to all numbers, and are very interesting and practical.

Nexen's Method of Extracting Higher Roots

The following method of extracting the cube and other high roots is easier than the regular method of the arithmetics, and is also of interest, because, by carrying out the same principles, any root may be extracted; examples are given of the extraction of the 11th root in Prof. Nexen's book. The cube root extraction follows here.

Point off the number in threes as usual and by inspection and trial if necessary get the nearest root of the first period pointed off. This root may be either too large or too small, but it must be the nearest.

Raise this approximate root to the power represented by half of the exponent or if the exponent is an odd number, raise it to the power of what may be called the "larger half" of the exponent. For a cube root, where the exponent is 3, the "larger half" is 2; for the fifth root it would be 3.

Divide the original number by this quantity; the quotient will be an approximation to the cube root sought. Add to it twice the original approximate root and divide by 3; this gives an average which is a still closer approximation to the correct root.

With this new approximate root repeat the operation and so *ad libitum,* until a close enough result is obtained. Two operations are close enough for all ordinary purposes.

The rule seems long, but the operation is very short as will be seen in the examples given here.

To extract the cube root of 250, proceed as follows: A mental trial shows that 6 is the nearest integral root of 250; it is raised to the second power in accordance with the third paragraph of the rule, and the number is divided by it. We have therefore:

$$6=36; \qquad \begin{array}{r} 36\,)\,250\,(\,6.94 \\ 216 \\ \hline 340 \\ 324 \\ \hline 160 \end{array}$$

The quotient, 6.94, is an approximation to the root sought for; it is added to twice the first approximate root,

which was 6, and the average of the three is squared and used as a new divisor, because it is a closer approximation to the true root than either of the others.

$$
\begin{array}{rl}
6\times 2= & 12 \\
 & 6.94 \\
\hline
 & 18.94
\end{array}
$$

average is 6.31 and $6.31^2 = 39.82$ the next divisor.

$$
\begin{array}{l}
39.82\ )\ 250.000\ (\ 6.2782 \\
\quad\ 238\,92 \\
\hline
\quad\ \&c.
\end{array}
\qquad
\begin{array}{rl}
6.31\times 2= & 12.62 \\
 & 6.278 \\
\hline
 & 18.898
\end{array}
$$

average is 6.299

and $6.299^2 = 39.677$ is the next divisor and the quotient of 250 divided by 39.677 is 6.300. Using this to get an average exactly as in the preceding cases, we have:

$$
\begin{array}{rl}
6.299\times 2= & 12.598 \\
 & 6.300 \\
\hline
 & 18.898
\end{array}
$$

average is 6.2993

If required we can go on as far as we desire. But comparing the three approximations so far obtained they are so close that it is not worth while to go further for any ordinary calculation. The three approximations with their cubes are the following:

6.31	whose	cube	is	251.23
6.3063	"	"	"	250.79
6.2993	"	"	"	249.96

We now will extract the cube root of 377. The cube of 6 is 216, the cube of 7 is 343; the cube of 8 is 512;

the nearest root is therefore 7. We now proceed as before, but the operations will be merely indicated.

$$7^2 = 49; \quad 49\,)\,377\,(\,7.69 \qquad 7 \times 2 = 14$$

$$\begin{array}{rr} 343 & 7.69 \\ \hline \&c. & 21.69 \end{array} \quad \text{average is } 7.23$$

$$7.23^2 = 52.2729; \quad 52.2729\,)\,377.00000\,(\,7.212$$

$$\begin{array}{r} 365\,9103 \\ \hline \&c. \end{array}$$

$$\begin{array}{rr} 7.23 \times 2 = & 14.26 \\ & 7.212 \\ \hline & 21.672 \end{array} \quad \text{average is } 7.224$$

$$7.224^2 = 52.186; \quad 52.186\,)\,377.0000\,(\,7.224$$

$$\begin{array}{r} 365\,302 \\ \hline \&c \end{array}$$

Here we have reached the root because the number, 377, divided by the square of 7.224 gives a quotient which is the same, namely, 7.224.

The approximation to the exact cube root attained at any step of the calculation can be judged by comparing the number whose square is used as the divisor, with the quotient given by dividing the original number by the square in question.

The above result is given by logarithms also.

To extract the fifth root of a number, the approximate fifth root is the basis for the first division. This is raised to the cube and the quantity whose root is to be extracted is divided by it. The quotient is the approximate square

of the fifth root sought for. So we take the average obtained by dividing the sum of three times the cube root of the first divisor and of twice the square root of the quotient by 5 and repeat the operation with this as a divisor. By repeating the process three or four times a very close approximation to the exact fifth root is obtained. The process of determining the fifth root of 399 is shown here in outline, operations being indicated and results given.

As the approximate fifth root we start with 3, and divide by its cube which is 27.

$3^3 = 27;\quad 399 \div 27 = 14.77;\qquad 3 \times 3 = 9$

$\sqrt{14.77} = 3.85\qquad 3.85 \times 2 = 7.7$

$$5\,)\,\underline{16.7}$$
$$3.34$$

$3.34^3 = 37.26;\quad 399 \div 37.26 = 10.7084;\quad 3.34 \times 3 = 10.02$

$\sqrt{10.7084} = 3.29\qquad 3.29 \times 2 = 6.58$

$$5\,)\,\underline{16.60}$$
$$3.32$$

$3.32^3 = 36.59;\quad 399 \div 36.59 = 10.9046;\quad 3.32 \times 3 = 9.96$

$\sqrt{10.9046} = 3.302\qquad 3.302 \times 2 = 6.604$

$$5\,)\,\underline{16.564}$$
$$3.3128$$

The fifth root given by logarithms is 3.3125, practically identical.

Extractions of the fifth root by regular arithmetical process is a very long and tedious operation. The above method can be carried out to any desired degree of accuracy.

This method can be used for the square root, although the regular method is, to the writer's mind, more satisfactory. Let the square root of 2981 be required.

As the first divisor we take 5, the approximate square root of 29, the first two-figure period of the number in question.

$$2981 \div 5 = 5962;$$

In arranging the addition for the average we must take into account the fact that the divisor, 5, was referred to the first period, so in addition we put the two figure 5's under each other exactly as if the rest of the quotient was a decimal fraction.

$$\begin{array}{r} 5 \\ 5\ 962 \\ \hline 2\,)\ 10\ 962 \\ \hline 5\ 481 \end{array}$$

We now do the second division.

$$2981 \div 548 = 544.$$

The average of the divisor and quotient $(548 + 544) \div 2 = 546$, which is the square root as far as the figures are concerned, but as there are four figures in the original number the decimal point must go after the second figure, so that the root is 54.6.

The regular calculation carried out to four places of decimals gives 54.5985, practically the same. By cutting off no decimals and carrying the operation through some more divisions the root could have been brought to any degree of accuracy desired.

The above methods are interesting and instructive and practical for the cube and high roots, but where a higher root with a prime number for the exponent is to be extracted it is best to do it by logarithms.

CHAPTER X

EXPONENTS

Exponents of Powers

If a number greater than unity is multiplied once by itself, it is said to be raised to the second power; if multiplied by itself twice, it is said to be raised to third power. The same nomenclature is carried out for all powers. This system is not inconsistent as it may seem on first sight. The numerical value of the power, expressed by the exponent, does not directly indicate the number of multiplications involved in raising the number to the given power; it indicates the number of times the number raised enters into the multiplication. To raise a number to the second power, as it is multiplied once by itself, it is obvious that it will enter twice into the multiplication; if raised to the third power it will enter three times and so on.

Any power of a number, such power being expressed by an integral positive number, contains the original number as factors as many times as the exponent of the power indicates.

The exponent of a power may be a positive or a negative integral or fractional number; if it is an improper fraction it may be written as a mixed number if desirable.

Suppose that the quantity, 2, is to be raised to the fifth power, whose exponent is 5. It must be multiplied four times by itself, thus: $2 \times 2 \times 2 \times 2 \times 2 = 32$.

Here there are four multiplications but five equal factors, each one the original number; the five factors determine the power as the fifth power, 2^5, in numerical value, 32.

Fractional Exponents

A fractional exponent with 1 for numerator indicates the root of the number corresponding to the denominator of the fraction.

Thus the number, 4, to the half power, in figures, $4^{1/2}$, means the square root of 4, which is 2; 81 to the one-fourth power, $81^{1/4}$, means the fourth root of 81, which is 3.

A fractional exponent may have any number for its numerator. The number affected by such an exponent, if the indicated operation is carried out, is to have the root indicated by the denominator of the fraction raised to the power expressed by the numerator. This may be put the other way, and the exponent may be taken as indicating the same root of the same power of the original number. Both statements mean the same in effect.

Thus 8 to the two-thirds power, $8^{2/3}$, means the cube root of the square of 8; the square of 8 is 64, and the cube root of 64 is 4, so $8^{2/3}=\sqrt[3]{8^2}=\sqrt[3]{64}=4$.

This could have been taken the other way. The cube root of 8 could have been raised to the square; the cube root of 8 is 2, and 2 raised to the square is 4.

The first process is generally the better one.

If the fractional exponent or index of the power is an improper fraction, the same method of treatment holds in all respects. But it is possible to introduce a somewhat different method by converting the improper

fraction into a mixed number. Before doing this a rule must be stated.

An exponent may consist of the sum of several numbers. To carry out the indicated operation, we may raise the number to the different indicated powers, multiply one by the other, and the continued product of all will give the value. Thus 4^{2+2+3} is the product of $4^2 \times 4^2 \times 4^3$ or $16 \times 16 \times 64 = 16{,}384$. We may add the exponents and raise the original number, 4, directly to the power indicated by their sum, which is 7; or $4^7 = 16{,}384$.

Returning to the consideration of a mixed number exponent, a mixed number is the sum of an integral number and a fraction. Therefore if we are to reduce a number with a mixed number exponent, we may separate the exponent into two parts, an integral and a fractional part, calculate each separately, and multiply them together.

What is the value of $9^{3/2}$?

$$9^{3/2} = 9^{1\frac{1}{2}} = 9^{1+\frac{1}{2}}, \text{ and } 9 \times 9^{\frac{1}{2}} = 9 \times 3 = 27.$$

Operating directly with the fractional exponent, we raise 9 to the cube giving 729, and extract the square root of 729, which gives 27 as before. The result is the same in both cases; there are two ways of doing the same thing.

We now come to another rule concerning exponents.

An exponent may be indicated by the subtraction of one number from another. In such case the original number may be raised to the power represented by the numerical difference between the exponents, with the sign, + or —, of the larger exponent prefixed to the difference. Thus 4^{3-1} may be expressed as 4^2; 4^{1-3} may

be expressed as 4^{-2}. But we may, if we wish, raise the number to each partial exponent, and divide the one with the positive exponent (the minuend) by the other.

Taking 4^{3-1} as before we may divide 4^3 by 4^1 or 64 by 4, giving 16; or we may say $4^2 = 16$, $4^{3-1} = 4^2$ and the answer, both results being identical.

Next take 4^{1-3}; following the rule we divide 4 by 4^3, which gives $\frac{4}{4^3} = \frac{4}{64} = \frac{1}{16}$, as before. And we may say

$4^{1-3} = 4^{-2} = \frac{1}{16}$ as before.

In the present book algebra has been avoided, as our object is to treat the subjects on a strictly arithmetical basis. A slight departure is inevitable here, as a negative exponent falls within the scope rather of algebra than of arithmetic.

Zero as an Exponent

Assume any number to have two equal exponents, one subtracted from the other. Such would be 2^{2-2}, 3^{3-3}, 9^{2-2}, and the like. We have seen that when a number has two exponents separated by the minus sign, as in these cases, the value of the expression is obtained by dividing the number raised to the first exponent by the same number raised to the second one. But here both exponents are the same; therefore the number raised to the power represented by one exponent will be the same as for the second exponent. Thus $2^{2-2} = 4 \div 4 = 1$. The same applies to 3^{3-3}, which gives $27 \div 27 = 1$, and $9^{2-2} = 81 \div 81 = 1$. Now we must note the value of the exponents of the numbers; the value of $2 - 2$ is 0;

that of $3 - 3$ is also 0. So we find that these numbers are of the zero or 0 power, and that any number of the 0 power is equal to unity.

Thus 5^0, 100^0, $\frac{1}{2}^0$ or any quantity affected with the exponent 0, is equal to 1.

This is in line with the converse, that unity is always equal to unity, whether an exponent is given it or not; $1^3 = 1$, $1^4 = 1$, and so for all exponents.

Prime Exponents

The following property of prime exponents is attributed to a Chinese source.

If 2 is raised to any power, whose exponent is a prime number, and if 2 is subtracted from it when so raised, the result will be divisible by the exponent used without a remainder; the quotient will be an integral number.

Take 3, a prime number, as the exponent of 2; this gives 8; from 8 subtract 2 and 6 is left, which is divisible by 3, the exponent used. The same may be tried with 5, also a prime number as the exponent. $2^5 = 32$, and $32 - 2 = 30$, which is divisible by the exponent used, which was 5.

Negative Exponents

A negative exponent indicates division of unity by the number with the negative exponent, after such number has been raised to the power indicated by the exponent. Thus 2^{-3} indicates division of unity by 2^3 or by 8, and putting it in fractional form we have $2^{-3} = \frac{1}{2^3}$ or what is the same thing, $\frac{1}{8}$.

The reason for this is the following:

We have $\frac{2^5}{2^2} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2} = \frac{2 \times 2 \times 2}{1} = 2^3$, or $2^5 \div 2^2 = 2^3$. Now by the same logic if $2^5 \div 2^2 = 2^3$, then $2^2 \div 2^5 = 2^{-3}$; but

$$2^2 \div 2^5 = \frac{2^2}{2^5}$$

and

$$\frac{2^2}{2^5} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2 \times 2 \times 2} = \frac{1}{2^3}.$$

Therefore $2^{-3} = \frac{1}{2^3}$.

Another rule refers to the division of an exponent. If an exponent of a number is divided by a number, the extraction of the root, expressed by the divisor, of the original quantity raised to the indicated power gives the value.

Suppose we have $4^{6 \div 2}$. We have: $4^6 = 4096$. Now as 6 is divided by 2, the square root of 4^6, which is 4096, must be extracted. $\sqrt{4096} = 64$.

It will be seen that the above can be done by fractional exponents, because $4^{6 \div 2} = 4^{6/2} = 4^3 = 64$.

The number 10, affected by exponents, positive, negative and fractional is used a great deal in electrical calculations, and is applicable in many other fields.

Powers of Ten

The units of electrical measurement are based on the centimeter, gram and second. They are far too large or

too small for practical use; accordingly a series of units called practical units have been adopted and are the only ones used in regular work. It is in the expression of the relation of these practical units to the C. G. S. units as the others are called, from the initials of centimeter, gram and second, that the number 10 affected by various coefficients is employed.

The practical unit of resistance, the ohm, is equal to 1,000,000,000 C. G. S. units of resistance. If the number 10 is multiplied by itself 8 times, it will enter as a factor into the operation, nine times; hence it will be raised to the ninth power, so that 10 with an exponent, 9, will represent the one thousand millions expressed above in figures; it is 10^9.

The exponent of any power of ten indicates the number of ciphers which come after the 1 in the numerical representation of that power. The above number contains nine ciphers to the right of the 1; therefore the number is the ninth power of ten and is expressed by the numeral ten with an exponent nine—10^9.

The volt is the practical unit of potential; it is equal to one hundred millions times the C. G. S. unit of potential; this factor is expressed by 10 with an exponent 8; it is 10^8.

The ampere is one tenth of a C. G. S. unit of current; this factor or multiplier is written with a negative exponent of 1, thus 10^{-1}, meaning $\frac{1}{10}$.

The unit of capacity is one one thousand millionth of th C. G. S. unit; this gives the multiplier a negative exponent of 9; it is 10^{-9}. But this practical unit is still inconveniently large; it is made one millionth this value by raising the negative exponent of ten to the—15th

power. But instead of having to write down a 1 followed by fifteen ciphers, all that has to be done is to change the exponent, so that the multiplier will be 10^{-15}.

When two or more numbers have identical exponents they can be multiplied by simply adding the exponents together and putting down the original number with the new exponent. This is rather an indication of multiplication than a real multiplication. Applying this rule to powers of ten we have such results as the following:

$$10^{10} \times 10^{3} = 10^{13}; \quad 10^{2} \times 10^{4} = 10^{6}; \quad 10^{7} \times 10^{8} = 10^{15}.$$

If the exponents have different signs the difference between them must be taken for the new exponent, and it is to have the sign of the larger numerical exponent affixed to it. Some examples follow:

$$10^{5} \times 10^{-3} = 10^{2}; \quad 10^{15} \times 10^{-10} = 10^{5}; \quad 10^{1} \times 10^{-2} = 10^{-1}.$$

To divide numbers with different exponents, but numerically identical, subtract the exponents algebraically, as in the following examples.

$$10^{2} \div 10^{3} = 10^{-1}; \quad 10^{5} \div 10^{3} = 10^{2}; \quad 10^{2} \div 10^{-4} = 10^{6}.$$

The subtraction is to be done algebraically; accordingly in the first and last example, following the laws of algebra, the sign of the subtrahend was changed and the two exponents were added; this constitutes algebraic subtraction. In the other cases the operation was done arithmetically.

CHAPTER XI

SQUARING THE CIRCLE

Squaring the Circle

A long historical treatise could be written on the subject of the relation between the diameter and circumference of the circle. The ancient term "Squaring the Circle" expresses one of the world's most famous problems, the finding of the length of the side of a square equal in area to a circle of a given diameter. To determine it, the one thing necessary to be known is the ratio of the diameter to the circumference of a circle. If that is known the rest is simple everyday arithmetic. So the term squaring the circle is rather antiquated.

One of the first things many of us were taught in the line of practical mathematics is that once around a circle is three times through or across it. This would be most convenient and is easy to remember, but has one trouble, it is not so. It is the merest approximation to the truth; it is so far from the truth that it should never be used.

Approximations of Former Times

In ancient times the Babylonians and the Jews are supposed to have used this incorrect figure, 3, as the multiple to produce the circumference from the diameter. The ancient Egyptians are credited with doing better than this, with the fractional expression $^{256}/_{81}$ which reduces to 3.1605, a very fair attempt at the value.

Archimedes held that it was less than $3\frac{1}{7}$ and more than $3\frac{10}{71}$; this puts it between 3.1428 and 3.1408, which is correct as far as it goes.

Ptolemy called it, in degree measurement, 3° 8′ 30″, reducing to $3 + \frac{8}{60} + \frac{30}{3600}$, or in decimal notation, 3.1416, which is very nearly right.

The Roman surveyors are supposed to have used 3 sometimes and at other times 4, and, when inclined to come a little nearer to the truth, to have used $3\frac{1}{8}$ as the factor.

To Asia is attributed the fraction, $\frac{49}{16}$ or 3.1416, and the quantity expressed as $\sqrt{10}$, which reduces to 3.1622, a very poor attempt at accuracy.

Metius' Value of π

In the year 1585 a celebrated mathematician, Metius, placed the missing number between $\frac{377}{170}$ and $\frac{333}{106}$ and, as it is surmised, perhaps by a lucky guess, found his famous fraction, $\frac{355}{113}$, giving the decimal 3.14159292, which is correct to six places. This is quite close enough for anything short of astronomical dimensions.

Shaw's Value

By the calculus a correct method of calculating it to any degree of accuracy, for it can never be exactly expressed, has been determined, and it has been carried out to most wonderful lengths by computators; one, William Shaw, carried it out to 707 places of decimals.

Geometrical Approximation

A very curious geometrical approximation is obtained by inscribing a square in a circle. If to three times the

diameter of the circle one fifth of the side of the square be added, it will give the circumference of the circle with an error of only $^{17}/_{100000}$. The diameter of the circle is equal to the diagonal of the square inscribed within it; the side of the square, taking its diagonal as 1, is the square root of one half, which is .7071 and this divided by 5 gives .1414, which is .002 less than the true decimal carried to the number of places indicated.

The letter by which the number in question is indicated is the Greek letter, π, pronounced *pi;* it is the Greek *p,* and is supposed to stand for perimeter.

Aid to Memorizing Pi

The following sentence is given as a memoria technica for the factor 3.1416 or π carried to the fourth decimal place. The number of letters in each word gives the digits of the factor.

3 1 4 1 6
Yes, I have a number.

Another rhyme by which to remember the value of π, carried out to the twelfth decimal place is the following; as before, the number of letters in each word gives the corresponding digit.

3 1 4 1 5 9
See I have a rhyme assisting
2 6 6 3 5 9 9
My feeble brains its tasks sometimes resisting.

The mathematical value is better than the grammatical.

Curious Determination of π by Probabilities

Buffon and Laplace determined that if a stick of a length less than the distance apart of a series of parallel lines was dropped upon the surface marked with such lines, the probability that it would fall across one of the lines is expressed by the formula $2L/\pi A$, in which L is the length of the stick, which must be less than the distance apart of the lines, or $L<A$, and A is the distance apart of the lines. Sometimes a board floor with boards of equal width is cited as the ruled area.

The experiment of throwing a stick on such a surface has been tried, with results curiously near the true value of π. In one case the stick was thrown 3,204 times and gave the value, 3.1553 for π; another trial of 600 throws gave 3.137 and in the third case, 1,120 throws gave 3.1419 for π.

The reader is referred to A. de Morgan's Budget of Paradoxes, London 1872; and in the Messenger of Mathematics, Cambridge, Eng. 1873, Vol. ii, pp. 113, 114.

Squaring the Circle Literally

Having the multiplier, 3.1415927...with which to obtain the length of the circumference from that of the diameter, we can square the circle; this is to find the side of the square equal in area to the circle of any given diameter. The area of a circle is equal to the product of the square of the radius by π. For a circle of the diameter, 1, this area is .7853982 . . . . The side of a square of this area is equal to the square root of the same number; this root is .8862. This number is the

length of the side of a square of the area of a circle of diameter, 1.

It will be sufficient to give the following as the value of π; it is far more accurate than will ever be required by our readers. It is: 3.1415926535 . . . .

The error here is about one-thirty billionth, not far from an error of sixteen feet in the distance of the sun from the earth.

CHAPTER XII

MISCELLANEOUS

Prime Numbers

A prime number is one which is indivisible except by 1 and by itself. 2, 3, 5, 7, 11 and 13 are prime numbers. Excepting the number 2 they cannot be even; no prime number can terminate in 5, except 5 itself; it follows that after the first period of ten is passed a prime number can only end with a 1, 3, 7 or 9.

Properties of Prime Numbers

A curious property of prime numbers is that, except 2 and 3, if any one of them is increased or diminished by 1, one of the results will be divisible by 6. 101 is a prime number. Diminished by 1 it gives 100, which is indivisible by 6. But if the 1 be added we have 102 which is divisible by 6, as it is the product of 6 and 17. Take next the prime number 73; increase it by 1 and we have 74, which is indivisible by 6, but subtracting 1 gives 72, which is divisible. Take 23, a prime number, add 1, and we have 24, which is divisible by 6.

The product of two prime numbers can never be a square number.

Beginning with 2 and ending with 9973 Hutton gives a list of the prime numbers between these limits; their total is 1139.

The largest prime number known up to the present time is expressed as $2^{61} - 1$; it contains 19 digits.

How to Find Prime Numbers

To find the prime numbers proceed as follows:

Write out the odd integral numbers, 3, 5, 7, 9 and so on as far as desired. Cross out every third number after 3, for none of these are prime. Then cross out in addition to these every fifth number after 5, and then every seventh number after seven and so on. It becomes excessively laborious after high numbers are reached.

If we write out the odd integrals, 3, 5, 7, 9, 11, 13, &c we will find that the numbers eliminated by counting every third one from 3, are 9, 15, 21, 27, 33 .... Those eliminated by counting every fifth one from 5 are 15 (already eliminated by 3), 25, 34,... This leaves as prime numbers 3, 5, 7, 11, 13, 17, 19, 23, 29, 31. If the seven elimination be applied it will only touch up to this point the numbers 21 and 35, both of which have been eliminated, one by the three elimination (21) and one by the five elimination (35).

Perfect Numbers

The aliquot parts of a number are factors or integral numbers which multiplied by other integers give the original number. The number 1 is included. Thus the number six has as its integral parts or numbers, 1, 2 and 3. These three numbers multiplied respectively by 6, by 3 and by 2 give 6 as the product in each case.

A number the sum of whose aliquot parts gives the number in question is called a perfect number. If the aliquot parts of 6 are added together, we have $1+2+3=6$; therefore 6 is a perfect number.

The next perfect number in the numerical order is 28,

Its aliquot parts are 1, 2, 4, 7 and 14. Each of these is an aliquot part, because it will give 28 when multiplied by the proper number. Another way to put it is to say that an aliquot part of a number is a factor of the number; it is any number, which can divide it without a remainder. Now if these five numbers are added together we will obtain the original number, 28; therefore 28 is a perfect number.

To find all the perfect numbers take the geometrical progression, 2, 4, 8, 16, 32, . . . carrying it as far as desired. From this series select those numbers, which diminished by unity give prime numbers. Such are 4, 8, 32, 128 and 8192, if the series is carried out far enough to give it. Diminished by unity these numbers become 3, 7, 31, 127, 8191. Each is now to be multiplied by the geometrical progression which preceded the one from which it is derived. Thus 3 is derived from 4 of the original series, and 4 is preceded by 2 in the series, so 3 is multiplied by 2, giving 6, the first of the perfect numbers. Following out the rule 7 is to be multiplied by 4, 31 is to be multiplied by 16 and so on as far as desired. The result is very meager for there are very few perfect numbers; up to 10^{11} Hutton gives only eight of them.

All perfect numbers or nearly all end in 6 or in 28.

Amicable Numbers

Amicable numbers are numbers so related to each other that the sum of the aliquot parts of one of them gives the other number.

If we take the number 220, its aliquot parts are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; the sum of these is 284. The aliquot parts of 284 are 1, 2, 4, 71 and 142,

and the sum of these is 220; therefore the two numbers, 220 and 284 are amicable numbers.

Other amicabie numbers are 17296 and 18416; after these come 9363584 and 9437056. It will be seen that they are excessively rare.

To calculate them start with the same geometrical progression as in the determination of perfect numbers. Call it series A. Multiply each term by 3 and write it under the one from which it was derived. Call this series B. Multiply each number of series B by the number preceding it in the same series and diminish it by unity. Write these down each under the one from which it was derived. Call this series C. The following are the series obtained.

Series A.	2	4	8	16	32	64......
Series B.	6	12	24	48	96	192......
Series C.		71	287	1151	4607	18431......
Series D.	5	11	23	47	95	191......

Subtract 1 from each figure of series B, and write it under the number from which it was derived. Cail this series D.

Take any prime number of series C, subject to this condition: the number under it in series D, and the number preceding this one in series D, must be prime numbers. Multiply the two numbers of series D together, and multiply their product by the number in series A directly over the larger one. Thus in series C, 71 is a prime number; so are 11 and 5 of series D. Following the rule we have $5 \times 11 \times 4 = 220$, one of the numbers. The other is obtained by multiplying 71 by the same number in series A; $71 \times 4 = 284$; this is the corresponding amicable number.

Law of the Square and the Cube

Everything else being equal the larger a cannon ball the farther will it go with the same initial velocity. The reason is based on what may be called the rule of the square and the cube. The resistance offered by the air to the passage of the projectile is due to the area of its cross-section and to what is called skin friction. The area of the section of two similar solids varies with the square of any corresponding linear dimension. This is the law in the case of all surfaces, plane or flat or curved as long as they are similar. A circle for instance varies in area with the square of its diameter. The cross-section of a cannonball is a circle so the same rule applies. The resistance to its flight due to the cross-section therefore varies with the square of its diameter. As regards skin friction the same rule applies; the area of the surface of the projectile varies also with the square of the diameter. Now the force which keeps the ball in motion is its inertia and this depends on its weight and is proportional thereto. But the weight of a body varies with its cubic contents and in the case of two solids of similar shape the cubic contents vary with the cube of any corresponding linear dimension. We may use the diameter; we have therefore the resistance varying with the square of the diameter and the resistance with the cube of the same. The cubes of any two numbers vary more than do the squares; therefore an increase of size will increase the weight more in proportion than it will increase the surface or the cross-section; the larger shell will go further than the smaller one as it is heavier in proportion to its surface.

Compare for instance a two inch and a three inch ball. The surface resistances are in the proportion of the

squares of the diameters; or as 4 is to 9. This is as 1 is to 2¼. The propelling force of inertia is in the ratio or proportion of the cubes, or as 8 is to 27; this is the ratio of 1 to 3⅜. The larger shell with only 2¼ times the resistance of skin friction and cross-section of that of the larger shell has 3⅜ times the propelling force of its inertia which as we know is in proportion to its weight. That is why a large projectile goes further than a small one.

This rule has many applications other than the one of projectiles. The force of gravity acts on all bodies in proportion to their weight; in a vacuum all bodies large or small fall with the same speed. In the air their fall is retarded by their skin friction and cross-sectional resistance as it may be called. Again the rule of the square and the cube comes into play and the larger body falls the faster. Dust falls with extreme slowness although it may be of the same material as a great rock which would fall with tremendous velocity.

The Four O'Clock Symbol

The clock faces, on which Roman numerals are used, it will be observed, have the four indicated by four I's, instead of by IV. It is said that the reason of this goes as far back as Charles the Fifth or as it is written Charles V. He is said to have declared that nothing should go before the V of his title, so he thus eliminated the usual IV, and there was nothing to do but to use four letter I's.

The Number 108 in Our Solar and Lunar Systems

The diameter of the sun is approximately 108 times that of the earth; the distance of the earth from the sun

is approximately 108 times the diameter of the sun. The distance from the moon to the earth is approximately 108 times the diameter of the moon.

Automobile Tires

For any size automobile wheel there is a specific tire, and besides this there is what is called an oversize tire. This is a tire of one half inch larger diameter of caliber than that of the regular size. The tire has to fit the same rim; the problem is what will be the increase in full diameter for the half inch increase in caliber.

As it has to fit the same rim the inner circle of the larger tire must be the same as that of the smaller one. The increased caliber adds a half inch all around the tire. Therefore as there is a half inch extra on every side of the tire it results in giving a tire one inch larger. Therefore the oversize for a 34×4 inch tire is $35 \times 4\frac{1}{2}$. Add an inch to the cross diameter of any given tire and a half inch to the caliber or as it may be termed to the small diameter and you will have the oversize.

If an automobile tire is priced at a certain amount the one of the next larger caliber, as a $4\frac{1}{2}$ and a 4 inch tire, seems always to be priced disproportionately high. But suppose we take a three inch tire to be compared with a three and a half inch one. If both were of identical thickness and of equally heavy construction the larger one would have 5/30ths extra material in its make-up. This is almost 17%, so on this basis alone it will be seen that there is a reason for a large increase in price. But an inch on the cross diameter although it sounds like more is much less; for two tires of 30 and 31 inch size

and of the same caliber the extra material in the larger one would be less than 3½% of the total.

The Two Clerks

Two clerks start work in the same office at the same salary—each gets $1,000 per annum. One has his pay increased $50 every six months; the salary of the other is raised $200 every year. The salaries are paid each half year. Which clerk fares the best?

At first sight it would seem that the clerk whose salary was raised $200 each year would get the most, but it will be found on calculating the annual returns in each case that the clerk receiving the $50 raise, every six months, keeps constantly ahead of the other. At the end of the first year the first clerk has received $1,050 and the other one $1,000; at the end of the second the first clerk will receive for his year of work $1,250; the other one, $1,200. It keeps on in the same way; the first clerk is always $50 ahead of the second one.

Wine and Water Paradox

Let a tumbler, *A,* be half full of water, and a second tumbler, *B,* be half full of wine. A teaspoonful of wine is poured from *B* into *A,* and then a teaspoonful of the mixed water and wine is taken out of *A* and is put into *B*. The question is; after all this, has the tumbler, *B,* lost more wine than the water, which has been lost by the tumbler *A* ?

Most people will say at once that more wine was lost by *B,* than the water lost by *A*; the correct answer is that they come out even, one losing less than a teaspoon-

ful of wine, the other the same quantity of water; each loses the same fraction of a spoonful.

Paradox of Squares of Fractions of a Number

A shepherd is to divide his flock in two unequal parts in such a way, that the square of the smaller part added to the larger part will be equal to the square of the larger part added to the smaller part. How does he do it?

This is an excellent catch and illustrates the difference between the powers of integral numbers and those of fractions. With integral numbers it is impossible, and the catch consists in doing it with fractions of the herd, dividing it into two fractions, each expressed as a fraction of the whole.

He may divide it into five eighths and three eighths, whose sum makes unity, and is taken as giving the whole flock. Performing the required operations, we have:

$$(5/8)^2 + 3/8 = (3/8)^2 + 5/8 = 49/64$$

The calculation may be made with decimal fractions, thus:

$$(.625)^2 + .375 = (.375)^2 + 675 = .765625.$$

Divining Numbers Thought of

The following is an interesting example of old time arithmetical puzzles. It is taken from Laurechon, going back to the 17th century.

Let someone think of three numbers; he is to double the first one, add 5, and multiply the sum by 5, and add

10; to this he is to add the second number, and multiply by 10; then he is to add the third number, and subtract 350; the remainder will be the three numbers in their proper order, as he entered them into the computation.

Suppose the numbers thought of were 2, 3 and 4. We then proceed as follows: $(2\times 2)+5=45$; $45+10=55$; $55+3=58$; $58\times 10=580$; $580+4=584$; and finally, $584-350=234$, which are the three numbers thought of in the order in which they were entered into the computation.

A Paradox of the Time Card

The following is a good illustration of confusion between two units having the same name but different values.

A clerk applied for a job in a business house and said he was worth $1,500 per annum. He was told he was not worth it.

Said the proprietor:

"There are 365 days in a year............	365
You sleep 8 hours a day (⅓ of a day)....	122
Days left	243
You rest 8 hours a day..................	122
Days left	121
There are 52 Sundays in the year........	52
Days left	69
You have one half day on Saturdays.....	26
Days left	43

You have 1½ hours for lunch............	28
Days left	15
You have two weeks' vacation..........	14
Days left	1

This is July 4th and we close, so that you do no work at all."

Remembering Telephone Numbers

It is generally unsafe to trust to the memory for too many telephone numbers. But there are in many cases peculiarities about them, which give a basis for retaining them in mind. Memoria Technica or Artificial Memory is the term describing such methods as we illustrate here.

Take the telephone number 2579. It is made up thus: the sum of the first two numbers, $2+5$, gives the third number, 7. The sum of the first and third number, $2+7$ gives the last number, 9.

Try the number 1140; the sum of the first two numbers gives half the third one.

Try 5292; the sum of the first, second and last number gives the third one.

Try 1121. The sum of the first two numbers gives the third number.

Try 190 and 187. 1 and 9 make 10, this gives the final cipher for 190; the sum of the first and last number gives the middle number of 187.

Try 6447. The sum of the two middle numbers gives 8; two greater than the first number and one greater than the last one.

Many numbers involve sequences or inverted sequences. Inversion may be partial or complete. Such are 4678 and 6876.

Making the Nine Digits in Proper Order Add up to One Hundred

It is required to combine the nine digits, keeping them in their proper order, so that they shall add up to 100. The trick, for it is little better, is done by combining multiplication of the last two, i. e. 9 and 8, with the addition of the first seven digits, thus:

$$(9 \times 8) + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 100$$

The same digits can be made to give the same sum in several other ways, none of any importance, as below:

$$15 + 36 + 47 = 98 + 2 = 100$$
$$(98 - 76) + 54 + 3 + 21 = 100$$

Curious Multiplication

The next series of operations brings out the nine digits in their natural and also in their inverted order. The calculation is self explanatory.

$$(1 \times 8) + 1 = 9$$
$$(12 \times 8) + 2 = 98$$
$$(123 \times 8) + 3 = 987$$
$$(1234 \times 8) + 4 = 9876$$
$$(12345 \times 8) + 5 = 98765$$
$$(123456 \times 8) + 6 = 987654$$
$$(1234567 \times 8) + 7 = 9876543$$
$$(12345678 \times 8) + 8 = 98765432$$
$$(123456789 \times 8) + 9 = 987654321$$

A Unique Number

The smallest number, of which the alternate figures are ciphers, and which is divisible by 9 and also by 11 is the following:

909090909090909090909

Dividing it by 9 gives a series of tens; dividing it by 11 gives:

82644628099173553719

The quotient by 11 is quite curious; it contains various sequences of identical numbers in direct and reversed order; they are 8 2 6 4 and 4 6 2 8, 1 7 3 5 5 and 5 5 3 7 1 or better, instead of the last two, 1 7 3 5 and 5371. Others can be worked out.

Curious Multiplication and Addition

The following set of combined multiplication and addition is somewhat similar to the example already given.

$$
\begin{aligned}
(1 \times 9) + 2 &= 11 \\
(12 \times 9) + 3 &= 111 \\
(123 \times 9) + 4 &= 1111 \\
(1234 \times 9) + 5 &= 11111 \\
(12345 \times 9) + 6 &= 111111 \\
(123456 \times 9) + 7 &= 1111111 \\
(1234567 \times 9) + 8 &= 11111111 \\
(12345678 \times 9) + 9 &= 111111111 \\
(123456789 \times 9) + 10 &= 1111111111
\end{aligned}
$$

Multiplications of Nine

If we multiply 9 by 21 the product is 189; if the multiplication is by 321, the product is 2889. If successive

multiplications are carried out with the inverted digits, one being added to the row each time, the successive multipliers will be 21, 321, (as above) 4321, 54321, and so on up to the full row of digits inverted, 987654321. The left hand figures of the successive products will be the digits in regular order, the last figure in each case will be 9, and there will be 8's between the first and last digits, the number of such 8's being one less than the first or left hand figure of the multiplier. The successive products thus will be: 189, 2889, (as above) 38889, 488889, 5888889, 68888889, 788888889, 8888888889, 98888888889.

Sometimes to make it more striking a subtraction of 1 from each of the products is prescribed; this gives final 8's instead of 9's.

The successive multiplications may be arranged in a sort of pyramidal form, similar to the sets of multiplications above.

If the digits, omitting 8, are written down in their regular order, we shall have a number, which, if multiplied by 9, will give a succession of units or 1's. The multiplication follows:

$$\begin{array}{r} 12345679 \\ 9 \\ \hline 111111111 \end{array}$$

One curious thing about this is the effect of omitting the 8; this omission gives the succession of 1's. If the 8 is not omitted, a cipher makes its appearance in the place below it. The multiplication follows:

$$\begin{array}{r} 123456789 \\ 9 \\ \hline 1111111101 \end{array}$$

The reader may try placing the nine digits in reverse order and multiplying them by 9.

The following is a curious set of multiplications or of squarings:

$$11 \times 11 = 121$$
$$111 \times 111 = 12321$$
$$1111 \times 1111 = 1234321$$
$$11111 \times 11111 = 123454321$$

The same operations can be carried out as far as desired, but the symmetry is impaired after the number of 1's exceeds nine.

Properties of Numbers

It is not easy to define the term "Properties of Numbers," but the conception of just what the term means is easily reached by examples. Thus one of the properties of the number one is that its square, cube and all other powers are equal each one to unity. Any number whose last two digits are divisible by four is divisible by four throughout. This may be taken as a property of the number four. A number is a perfect number if the sum of all its factors is equal to itself. Thus the factors of 6 are 1, 2 and 3, and the sum of these three figures is the original number 6. One of the properties of 6 then is that it is a perfect number. The factors of 28 are 1, 2, 4, 7, and 14; add these together and you will have 28. Therefore one of the properties of 28 is that it is a perfect number.

Properties of Nine

The number nine exceeds in interesting and practical properties all other digits. It is a wonder that it has not been more popular with the tribe of astrologers and others of that class, as a lucky or unlucky number, instead of three or seven. Even the superstitious crap player neglects it.

If 9 is used as the multiplier of any number from 1 to 20 the sum of the digits or figures of the product will be 9. Thus $13 \times 9 = 117$, and $1 + 1 + 7 = 9$. Or $18 \times 9 = 162$, and as before $1 + 6 + 2 = 9$.

Now write down the series of products of 9 by all the numbers from 1 to 20 and see if there is not something peculiar in this series of properties. Thus $9 \times 2 = 18$ and $9 \times 9 = 81$; each of these products is composed of the same digits but in reverse order. Try 9 multiplied by 13 and by 19; the products are respectively 117 and 171, the same digits but the last two are reversed. The numbers 10, 11 and 16 are the only multipliers which do not pair with any others as above, and if the cipher, 0, is left out of account, the refractory numbers reduce to 11 and 16; their products by 9 are respectively 99 and 144. A moment's inspection shows that they are incapable of reversing in the way all the other products are; hence they fall out by the wayside. Coming back to our paired numbers we will find that the sum of the multipliers for each pair under 10 times 9 will be the same, namely 11. Thus $2 \times 9 = 18$ pairs off with $9 \times 9 = 81$; $4 \times 9 = 36$ pairs off with $7 \times 9 = 63$. The two pairs of multipliers are 2 and 9 whose sum is 11, and 4 and 7 whose sum is also 11. This applies to single digit multipliers; now if we try the same with multipliers from

12 to 20, the sum of each pair of multipliers is 32. $13 \times 9 = 117$ and $19 \times 9 = 171$ are such a pair, and the sum of the multipliers is 32, thus $13 + 19 = 32$. After the multiplier 20 another pair comes in 21 and 22, and then another set from 23 to 30; this set and other succeeding sets may be left to the reader's investigations.

The Bookkeeper's Error

Suppose a bookkeeper finds a persistent difference in his trial balance. One of the properties of nine is that if any number is reversed or has its digits transposed, the difference between the two numbers will be divisible by 9 without a remainder. Thus transpose the digits of 279; this gives 972. The difference is 693, for $972 - 279 = 693$, and this difference is divisible by 9 without any remainder. If therefore the unhappy bookkeeper's difference is divisible by 9, it is almost certain that he has transposed some figures. This transposition of figures is a frequent source of trouble in bookkeeping, and the division by 9 without a remainder facilitates the finding of the error, when such error is due to transposition. Assume in the two columns of additions given below that the right hand one is correct, but that the left hand one is what the accountant really did. His erroneous inversion is put in italics.

6.35	6.35	173.21	173.21
1599.*75*	1599.*57*	1*87*.26	1*78*.26
381.23	381.23		
		360.47	351.47
1987.33	1987.15	and $1987.33 - 1987.15 = 18$;	

and $1987.33 - 1987.15 = 18$; the divisibility of the

difference 18 by 9 without a remainder indicates the fact that two figures were transposed and makes the location of the error easier than it would be without this clue. The rule is not infallible; a difference divisible by 9 might exist and not be due to an inversion, but it is a case of probability; and inversion or transposition probably was the cause of the trouble. In the second pair the error is in the middle of the number but the same rule holds. It also holds for the transposition of several figures.

A Mystery in Money

Take any sum of money less than ten dollars ($10.00) transpose the digits and take the difference between the two sums thus obtained. Then to the difference add its own transposition and you will always get the same amount, namely $10.89. Try it and see. Here are a few examples.

	$6.73	$9.91	$2.31	$0.01
	3.76	1.99	1.32	1.00
dif.	2.97	7.92	0.99	0.99
	7.92	2.97	9.90	9.90
sum	$10.89	$10.89	$10.89	$10.89

Of course the dollars have nothing to do with it; they merely make it more picturesque and if it is used as a sort of trick or magic the dollar sign obscures it a little, a desirable feature in parlor magic and the like.

Thus a person is told to write down a sum of money less than ten dollars, and whose first and last figures must be different. Then he is to write down the trans-

posed figures directly under it. Ask him what the first or last figure is. As the sum of the first and last figures is always nine and the central figure is always nine you at once tell him what the number is. Then he is told to transpose this last number, writing it under the other, and to add the two. Then he is told what the result is, namely $10.98. To make it a little more mysterious it is well to tell him to put any number you name below the lower pair of numbers and then to add them. All you have to do is to add this to 1,089 and tell him the result. The object of this is to prevent the repeated production of 1,089. The examples below illustrate this. In one case you can tell him to add 25, in the other case to add 31.

1.98	2.29
8.91	9.22
———	———
6.93	6.93
3.96	3.96
25	31
———	———
11.14	11.20

A professional magician as a rule avoids performing the same trick too often, so it is well to vary this one by sometimes not giving your friend the third number, but to let him finish the operation before telling the number found. It is well always to add a different number as just described, perhaps just once to omit the addition and telling him that he has ten dollars and eighty-nine cents as the result of his figuring. After that always add in some number and always a different one.

But while the dollars and cents are only a sort of concealing device in the above, it can be done with other denominate numbers so as to be much more puzzling. To do it with pounds, shillings and pence, the pounds must not exceed 12, and the pounds and pence must differ in number. The result will always be £12 18s. 11d. We will do it with £12 9s. 6d. and with £3 7s. 2d.

£12	9s.	6d.		£ 3	7s.	2d.
6	9	12		2	7	3
5	19	6		0	19	11
6	19	5		11	19	0
12	18	11		12	18	11

Divining a Sum of Digits

Another interesting bit of mystification may be based on properties of nine. Take any transposable number, that is to say any number whose first and last digits differ, and square it. Transpose it and square the result. Then on subtracting one from the other the result will be a number without any excess of nines. Thus a person is told to think of a number; then to square the number; then to transpose and square the transposed one; to subtract one from the other and to add the digits of the difference and to tell you the first or the last digit of this sum. At once you tell him the number which the addition of the digits has given. All you have to do is to annex to the number given a number which will make it a multiple of nine. If he says the first number is 1, the whole sum of the digits is 18; if 2 it is 27.

If you are asked to do it again introduce a slight variation. Let any two transposable pairs of numbers be multiplied together; then transpose one of them and multiply the original by the transposed one. Subtract one product from the other and again the result will be as before; the sum of the digits will be nine or a multiple thereof.

An example by the *first* method is given here.

$$3391 \times 3391 = 11498881$$
$$1933 \times 1933 = \underline{\ 3736489}$$

Subtracting 7762392 whose digits added give 36 or 9×4

The *second* method may come next.

$$69 \times 69 = 4761$$
$$69 \times 96 = \underline{6624}$$

1863 whose digital sum is 18 or 9×2.

When in the first case you are told that 3 is the first digit of the sum, you simply annex 6 to make 36; in the second case when told that 1 is the first number you annex 8 to make 18; this gives 36 and 18, both multiples of 9.

Such a number as 1,981 can be used because the two figures in the middle are reversible and carry out the law.

Other Mystifications

Take any number, invert it and subtract the smaller of the two from the larger; multiply by any number and

then cross out a number and give the number left; the digits are to be read in their order as a regular number. If this number is subtracted from the next highest multiple of nine, the difference between it and the next highest multiple of nine will be the number crossed out. Take the numbers 1,293, 146 and 97.

1293	146	97
3921	641	79
2628	495	18
Cross out 6; 228 is left. The next highest multiple of 9 is 234, and 234 — 228 = 6, the number crossed out.	Cross out 5; which leaves 49 — and we have 54—49= 5.	Cross out 8, leaving 10; this gives 18—10=8.

If any of the differences had been multiplied it would have made no difference; it may be assumed that they are multiplied by 1. Thus multiply the first difference, 2,628 by 3; this gives 7884; cross out one of the 8's leaving 784; the next highest multiple of 9 is $9 \times 88 = 792$ and the difference is $792 - 784 = 8$, which was the number crossed out.

Take any two numbers less than 10; multiply one of them by 5, add 7, multiply by 2, add the other number, subtract 14; the number resulting will be made up of the digits of the two numbers.

Let the two numbers be 3 and 6. The successive operations are these: $3 \times 5 = 15$; $15 + 7 = 22$; 22×2

$=44$; $44+6=50$; $50-14=36$, which are the two digits we started with.

Next try the same operation with numbers of two digits, say 17 and 28. Carrying out the operations exactly as before gives as the final result, 198. This seems to be wrong. But if the middle digit is split up it can give $7+2=9$. We therefore read 198 as 17 and 28, which are the two numbers we started with. This can be done with all two digit numbers. The dividing the middle digit into two gives the original numbers.

The same sort of operation can be applied to three single numbers. Multiply the first number by 10 and add the second; multiply the sum by 10 and add the third number; the three numbers will then appear in their proper order.

Take the numbers 2, 4 and 7. Carrying out the operation gives $(2\times 10)+4=24$; $(24\times 10)+7=247$; the three numbers we started with in their proper order.

If this operation be followed out, it will be seen that nothing has been done except to multiply the first of the three numbers by 100, the second by 10 and the last is not multiplied by anything. These multiplications throw the three numbers into the hundreds, tens and units places in the final number.

The above principle can be applied to the investigation of the inversion operation just given. It gives a general formula for all cases. Call the three digits of the number to be treated a, b and c. Writing them as numbers, on the basis of the multiplications as above, gives $100a+10b+c$. Writing this out again and placing beneath it its inversion, remembering that units now become hundreds and hundreds become units, and subtracting we have:

$$\begin{array}{l} 100a + 10b + c \\ 100c + 10b + a \\ \hline 99a \quad - 99c = 99(a - c) \end{array}$$

Suppose our number is 795; then $a - c$ is $7 - 5 = 2$, and $99 \times 2 = 198$ is the number which will be obtained by inverting and subtracting.

Dividing Multiples of Ten by Eleven

If 20 is divided by 11 the quotient is 1.81818. . . . If 30 is to be divided simply increase the first number of the last quotient by 1 and decrease the second number also by 1 and repeat the pair as long as you wish. The same rule applies all down the line, and the results of dividing multiples of 10 and 11 are tabulated here.

$20 \div 11 = 1.81818....$	$60 \div 11 = 5.45454....$
$30 \div 11 = 2.72727....$	$70 \div 11 = 6.36363....$
$40 \div 11 = 3.63636....$	$80 \div 11 = 7.27272....$
$50 \div 11 = 4.54545....$	$90 \div 11 = 8.18181....$

Inspection will show perfectly how the rule of adding and subtracting 1 is carried out. Another thing to be observed is that each pair or couple of digits in the quotients above is divisible by 9; it makes no difference what pair of numbers are taken as long as they are next to one another in any of the quotients. From the first quotient of the tabulation we get the pairs 18 and 81, both divisible by 9, and the same applies to all the other quotients.

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